

SOME QUALITATIVE PROPERTIES OF THE SOLUTIONS OF THE MAGNETOHYDRODYNAMIC EQUATIONS FOR NONLINEAR BIPOLAR FLUIDS

PAUL ANDRÉ RAZAFIMANDIMBY

ABSTRACT. In this article we study the long-time behaviour of a system of nonlinear Partial Differential Equations (PDEs) modelling the motion of incompressible, isothermal and conducting modified bipolar fluids in presence of magnetic field. We mainly prove the existence of a global attractor denoted by \mathcal{A} for the nonlinear semigroup associated to the aforementioned systems of nonlinear PDEs. We also show that this nonlinear semigroup is uniformly differentiable on \mathcal{A} . This fact enables us to go further and prove that the attractor \mathcal{A} is of finite-dimensional and we give an explicit bounds for its Hausdorff and fractal dimensions.

1. INTRODUCTION

Magnetohydrodynamics (MHD) is a branch of continuum mechanics which studies the motion of conducting fluids in the presence of magnetic fields. The system of Partial Differential Equations in MHD are basically obtained through the coupling of the dynamical equations of the fluids with the Maxwell's equations which is used to take into account the effect of the Lorentz force due to the magnetic field (see for example [9]). They play a fundamental role in Astrophysics, Geophysics, Plasma Physics, and in many other areas in applied sciences. In many of these, the MHD flow exhibits a turbulent behavior which is amongst the very challenging problems in nonlinear science. Due to the folklore fact that the Navier-Stokes is an accurate model for the motion of incompressible and turbulence in many practical situation, most of scientists have assumed that the fluids in these MHD equations follow the Newtonian law in which the reduced stress tensor $\mathbf{T}(\mathcal{E}(\mathbf{u}))$ is a linear function of the strain rate $\mathcal{E}(\mathbf{u}) = \frac{1}{2} (\mathbf{L} + \mathbf{L}^T)$, $\mathbf{L} = \nabla \mathbf{u}$, where \mathbf{u} is the velocity of the fluids. In this way we obtain the conventional system for MHD equations which has been the object of intensive mathematical research since the pioneering work of Ladyzhenskaya and Solonnikov [21]. We only cite [39], [38], [17], [13] for few relevant examples; the reader can consult [18] for a recent and detailed review.

There are two major problems that arise when studying MHD equations or hydrodynamic equations. First, it is well known that the uniqueness of weak solution of the three-dimensional NSE and MHD equations is a very difficult problem. As it is always not possible to prove the existence of global attractor in the case of lack uniqueness of solution, this is an obstacle for the investigation of the long-time behavior which is very

Date: March 7, 2013.

2000 Mathematics Subject Classification. 76W05, 35D30, 35B40, 35B41, 35K55.

Key words and phrases. Non-Newtonian fluids, Bipolar fluids, Shear thinning fluids, MHD, Magnetohydrodynamics, Asymptotic behavior, Long-time behavior, Global attractor, Hausdorff Dimension, Fractal Dimension.

important for the understanding of some physical features (such as the turbulence in hydrodynamics) of the models. We refer, for instance, to [1], [33], and [41] for some results in this direction for the case of autonomous Navier-Stokes and other autonomous equations of mathematical physics. Second, there are a lot of conducting materials that cannot be characterized by Newtonian fluids. To overcome these two problems one generally has to use other model of fluids or some perturbation of the Newtonian law. This has motivated scientists to consider (conducting) fluids models that allow \mathbf{T} being a nonlinear function of $\mathcal{E}(\mathbf{u})$. Fluids in the latter class are called Non-Newtonian fluids. We refer for example to the introduction of Biskamp's book [7] for some examples of these Non-Newtonian conducting fluids. In [23] and [24], Ladyzhenskaya considered a model of nonlinear fluids whose reduced stress tensor $\mathbf{T}(\mathcal{E}(\mathbf{u}))$ satisfies

$$\mathbf{T}(\mathcal{E}(\mathbf{u})) = 2(\varepsilon + \mu_0|\nabla\mathbf{u}|^r)\frac{\partial\mathbf{u}_i}{\partial x_j}, \quad r > 0.$$

Since then this model has been the object of intense mathematical analysis which have generated several important results. We refer to [14], [26] for some relevant examples. In [14] the authors emphasized important reasons for considering such model. Despite its mathematical success the Ladyzhenskaya model has received a lot of negative criticisms from physicists. Indeed the Ladyzhenskaya model is a mathematical model used to overcome the lack of uniqueness for the NSE; it does not have really a physical meaning as it does not satisfy some basic principles of continuum mechanics (the frame indifference principle) and thermodynamics (the Clausius-Duhem inequality). Necas, Novotny and Silhavy [31], Bellout, Bloom and Necas [2] has developed the theory of multipolar viscous fluids which was based on the earlier work of Necas and Silhavy [30]. Their theory is compatible with the basic principles of thermodynamics such as the Clausius-Duhem inequality and the principle of frame indifference, and their results up to date indicate that the theory of multipolar fluids may lead to a better understanding of hydrodynamic turbulence (see for example [5]). Bipolar fluids whose reduced stress tensor $\mathbf{T}(\mathcal{E}(\mathbf{u}))$ is defined by

$$\mathbf{T}(\mathcal{E}(\mathbf{u})) = 2\mu_0(\varepsilon + |\mathcal{E}(\mathbf{u})|^2)^{-\frac{\alpha}{2}}\mathcal{E}(\mathbf{u}) - 2\mu_1\Delta\mathcal{E}(\mathbf{u}), \quad (1)$$

form a particular class of multipolar fluids. If $0 \leq \alpha < 1$ then the fluids are said shear thinning, and shear thickening when $\alpha < 0$.

Let Ω be a simply-connected, and bounded domain of \mathbb{R}^n ($n = 2, 3$) such that the boundary $\partial\Omega$ is of class C^∞ . This will ensure the existence of a normal vector \mathbf{n} at each of its point. In this article we are aiming to the long-time behaviour of the system of nonlinear PDEs representing the motion of a conducting nonlinear bipolar fluids in presence of magnetic fields. This system is basically given by

$$\begin{cases} \frac{\partial\mathbf{u}}{\partial t} - \operatorname{div}\mathbf{T} + \mathbf{u} \cdot \nabla\mathbf{u} + \mu\mathbf{b} \times \operatorname{curl}\mathbf{b} + \nabla P = f & \text{in } \Omega \times (0, \infty), \\ \frac{\partial\mathbf{b}}{\partial t} + (S \operatorname{curl}\operatorname{curl}\mathbf{b} + \mu\mathbf{u} \cdot \nabla\mathbf{b} - \mu\mathbf{b} \cdot \nabla\mathbf{u}) = 0 & \text{in } \Omega \times (0, \infty) \\ \operatorname{div}\mathbf{u} = \operatorname{div}\mathbf{b} = 0 & \text{in } \Omega \times [0, \infty), \\ \mathbf{u} = \tau_{ijl}\mathbf{n}_j\mathbf{n}_l = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \mathbf{b} \cdot \mathbf{n} = \operatorname{curl}\mathbf{b} \times \mathbf{n} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ \mathbf{u}(0) = \mathbf{u}_0, \quad \mathbf{b}(0) = \mathbf{b}_0 & \text{in } \Omega, \end{cases} \quad (2)$$

where $\mathbf{u} = (u_i; i = 1, \dots, n)$, $\mathbf{b} = (b_i; i = 1, \dots, n)$ and P are functions defined on $\Omega \times [0, T]$, representing, respectively, the fluid velocity, the magnetic field and the modified pressure, at each point of $\Omega \times [0, \infty]$. S and μ are positive constants depending on the Reynolds

numbers of the fluid and magnetic fields, and the Hartman number. The quantities \mathbf{u}_0 and \mathbf{b}_0 are given initial velocity and magnetic field, respectively. The vector \mathbf{n} represent the normal to $\partial\Omega$ and τ_{ijl} is defined by

$$\tau_{ijl} = \frac{\partial \mathcal{E}_{ij}(\mathbf{u})}{\partial x_l}, i, j, l \in \{1, \dots, n\}.$$

Finally, \mathbf{T} designates the extra stress tensor of the Non-Newtonian fluid which is defined by (1). We suppose throughout that $\varepsilon, \mu_0, \mu_1$ are positive constants and $\alpha \in [0, 1)$. Hereafter we set

$$\Gamma(\mathbf{u}) = \mu_0 (\varepsilon + |\mathcal{E}(\mathbf{u})|^2)^{-\frac{\alpha}{2}}.$$

The structure of the nonlinearity of problem (2) makes it as interesting as any nonlinear evolution equations of mathematical physics such as the conventional MHD or the Navier-Stokes equations.

When $\mathbf{b} \equiv 0$ then (2) reduces to the PDEs describing the motion of isothermal incompressible nonlinear bipolar fluids which has been thoroughly investigated during the last two decades. Existence and uniqueness results of weak solution for nonlinear bipolar fluids with homogeneous boundary condition were given in [3], [4]. Existence of unique solution and asymptotic stability of the solutions for the case for nonhomogeneous boundary condition were investigated in [6]. The long-time behaviour of (2) with $\mathbf{b} \equiv 0$ is investigated in [5] for $\alpha \in [0, 1)$ and in [8] for $\alpha \in (-\infty, 0)$. Both of the authors of [5] and [8] proved the existence of a global attractor; they also show that the flow is finite-dimensional by giving explicit bounds for the Hausdorff and fractal dimensions of the global attractor. Existence results for multipolar fluids are also given in [27] and [31]. These are just examples of relevant work related to (2) with $\mathbf{b} \equiv 0$, the reader can consult [29] and [16] for detailed and recent reviews of results for nonlinear bipolar fluids.

For $\alpha = 0$, $\mu_1 = 0$, (2) is reduced to MHD equations which has been the object of intensive mathematical research since the pioneering work of Ladyzhenskaya and Solonnikov [21]. We only cite [39], [38], [17], [13] for few relevant examples; the reader can consult [18] for a recent and detailed review. Assuming that $\mu_1 = 0$ Samokhin studied the MHD equations arising from the coupling of the Ladyzhenskaya model with the Maxwell equations in [37], [36], [34], and [35]. In these papers he proved the existence of weak solution of the model for $\alpha \leq 1 - \frac{2n}{n+2}$. Later on Gunzburger and his collaborators (see [19] and [20]) generalized the settings of Samokhin by taking a fluid with a stress tensor having a more general structure. The authors of the last two papers analyzed the well-posedness and the control of (2) still in the case where $\mu_0 = 0$ and $\alpha \leq 1 - \frac{2n}{n+2}$.

To the best of our knowledge the preprint [32] is the only work treating the case $\{\mu_1 \neq 0, \mathbf{b} \neq 0, \alpha \in [1 - \frac{2n}{n+2}, 1)\}$. It was proved in [32] that if $(\mathbf{u}_0; \mathbf{b}_0) \in \mathbb{H}$ and $f \in \mathbb{V}^*$, then (2) with $\varepsilon = 1$ (but the author's result is still true for any $\varepsilon > 0$) has at least a global weak solution. This means that there exists a couple $(\mathbf{u}; \mathbf{b})$ such that

- $(\mathbf{u}; \mathbf{b}) \in L_{loc}^\infty(0, \infty; \mathbb{V}) \cap L_{loc}^2(0, \infty; \mathbb{H})$,
- $(\mathbf{u}; \mathbf{b})$ satisfies

$$\begin{aligned} \left(\frac{\partial \mathbf{u}(t)}{\partial t}, \phi \right) + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\phi)}{\partial x_k} \right) + (\Gamma(\mathbf{u}(t))\mathcal{E}(\mathbf{u}), \mathcal{E}(\phi)) + (\mathbf{u}(t) \cdot \nabla \mathbf{u}(t), \phi) \\ - \mu (\mathbf{b}(t) \cdot \nabla \mathbf{b}(t), \phi) = (f, \phi), \quad (3) \end{aligned}$$

and

$$\left(\frac{\partial \mathbf{b}(t)}{\partial t}, \psi \right) - S(\Delta \mathbf{b}(t), \psi) + \mu(\mathbf{u}(t) \cdot \nabla \mathbf{b}(t), \psi) - \mu(\mathbf{b}(t) \cdot \nabla \mathbf{u}(t), \psi) = 0, \quad (4)$$

for any couple $\Phi = (\phi; \psi)$ of smooth solenoidal functions and $t \in [0, T]$, $T > 0$. In (3) and throughout this work summations over repeated indices are enforced.

The author also analyzed the long-time behaviour of solutions of (2) with $\alpha \in [1 - \frac{2n}{n+2}, 1)$. He mainly proved the existence of trajectory attractor for the translation semigroup acting on the trajectories of the set of weak solutions and that of external forces. In the case when $\alpha \in [1 - \frac{2n}{n+2}, 1)$ we are not sure of the uniqueness of the weak solution, therefore the author of [32] follows the method in [10] (see also, [11] and [12]).

For $0 \leq \alpha < 1$ (which correspond to $1 < p \leq 2$ in the paper [32]) it is not difficult to check that the weak solution is unique. We will formally check this statement later on. This fact allows to define the nonlinear semigroup $\{\mathbb{S}(t); t \geq 0\}$ of the solution of (2). It is our purpose to study the long time behavior of this semigroup. The semigroup is formally defined by “let $t > 0$ to each initial value $(\mathbf{u}_0; \mathbf{b}_0) \in \mathbb{H}$ we associate an element $\mathbb{S}(t)(\mathbf{u}_0; \mathbf{b}_0) = (\mathbf{u}(t); \mathbf{b}(t))$ of \mathbb{H} which is the unique weak solution of (2).” Our results are as follow:

- (1) We give the existence of a compact global attractor \mathcal{A} for $\mathbb{S}(t)$. For this purpose we need to find absorbing sets in \mathbb{H} and in \mathbb{V} . Since \mathbb{V} is compact in \mathbb{H} then we can define from classical argument the existence of the compact global attractor \mathcal{A} .
- (2) We show that the semigroup $\mathbb{S}(t)$ is differentiable with respect to the initial data $(\mathbf{u}_0; \mathbf{b}_0) \in \mathcal{A}$. This result is crucial for establishing the next result.
- (3) We give an estimate for the bounds of the Hausdorff and fractal dimensions of \mathcal{A} .

To establish these results we mainly follow the idea in [5] and the classical results for the investigation of long-time behaviour of dissipative PDEs presented in [1], [33] and [41] for example. We should mention from the very beginning that every calculation we performed is formal, but we can check them rigourously by using the Galerkin approximation and pass to the limit as it was done in [32]. Also, even if we drew our inspiration from [1], [5], [15], [33], [38] and [41] the problem we treated here does not fall in the framework of these main references. Besides the usual nonlinear terms of the conventional MHD equations it contains another nonlinear term which exhibits the non-linear relationships between the reduced stress and the rate of strain $\mathcal{E}(\mathbf{u})$ of the conducting fluids. Because of this, the analysis of the behavior of the MHD model (2) tends to be much more complicated and subtle than that of the Newtonian MHD equations and the model in [5]. Hence, we have had to invest much effort to prove many important results which do not follow from the analysis in the aforementioned papers.

The organization of this article is as follows. We introduce the necessary notations for the mathematical theory of (2) in the next section. In Section 3 we prove the existence of absorbing sets in \mathbb{H} and in \mathbb{V} which enables us to show the existence of a global attractor for the nonlinear semigroup of solutions to (2). The uniform differentiability of $\mathbb{S}(t)$ on \mathcal{A} is established in Section 4. In the last section we give explicit bounds for the fractal dimension of the global attractor \mathcal{A} .

2. PRELIMINARY

We introduce some notations and background following the mathematical theory of hydrodynamics (see for instance [40]). For any $p \in [1, \infty)$, $\mathbb{L}^p(\Omega)$ and $\mathbb{W}^{m,p}(\Omega)$ are the spaces of functions taking values in \mathbb{R}^n such that each component belongs to the Lebesgue space $L^p(\Omega)$ and the Sobolev spaces $W^{m,p}(\Omega)$, respectively. For $p = 2$ we use $\mathbb{H}^m(\Omega)$ to describe $\mathbb{W}^{m,2}(\Omega)$. The symbols $|\cdot|$ and (\cdot, \cdot) are the \mathbb{L}^2 -norm and \mathbb{L}^2 -inner product, respectively. The norm of $\mathbb{W}^{2,m}(\Omega)$ (resp., $\mathbb{W}^{p,m}$) is denoted by $\|\cdot\|_m$ (resp., $\|\cdot\|_{p,m}$). As usual, $\mathbb{C}_0^\infty(\Omega)$ is the space of infinitely differentiable functions having compact support contained in Ω . The space $\mathbb{W}_0^{p,m}$ is the closure of $\mathbb{C}_0^\infty(\Omega)$ in $\mathbb{W}^{p,m}$. Now we introduce the following spaces

$$\begin{aligned}\mathcal{V}_1 &= \{\mathbf{u} \in \mathbb{C}_0^\infty(\Omega) : \operatorname{div} \mathbf{u} = 0\}, \\ \mathbb{H}_1 &= \{\mathbf{u} \in \mathbb{L}^2(\Omega) : \operatorname{div} \mathbf{u} = 0, \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathbb{V}_1 &= \left\{ \mathbf{u} \in \mathbb{H}^2(\Omega) : \operatorname{div} \mathbf{u} = 0, \mathbf{u} = \frac{\partial \mathbf{u}}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega \right\}.\end{aligned}$$

We also set

$$\begin{aligned}\mathcal{V}_2 &= \{\mathbf{b} \in \mathbb{C}_0^\infty(\Omega) : \operatorname{div} \mathbf{b} = 0; \mathbf{b} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathbb{H}_2 &= \text{the closure of } \mathcal{V}_2 \text{ in } \mathbb{L}^2(\Omega), \\ \mathbb{V}_2 &= \{\mathbf{b} \in \mathbb{H}^1(\Omega) : \operatorname{div} \mathbf{b} = 0; \mathbf{b} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}, \\ \mathbb{V}_3 &= \{\mathbf{b} \in \mathbb{H}^2(\Omega) : \operatorname{div} \mathbf{b} = 0; \mathbf{b} \cdot \mathbf{n} = \operatorname{curl} \mathbf{b} \times \mathbf{n} = 0 \text{ on } \partial\Omega\}\end{aligned}$$

Note that

$$\mathbb{H}_1 = \mathbb{H}_2.$$

We equip the space \mathbb{V}_1 with the norm $\|\cdot\|_2$ generated by the usual $\mathbb{H}^2(\Omega)$ -scalar product.

On \mathbb{V}_2 we define the scalar product

$$((\mathbf{u}, \mathbf{v}))_{\mathbb{V}_2} = (\operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}),$$

which coincides with the usual scalar product of $\mathbb{H}^1(\Omega)$. Hence, from now on $\|\mathbf{b}\|_1$ we will denote the norm of $\mathbf{b} \in \mathbb{V}_2$ in $\mathbb{H}^1(\Omega)$ and $\|\mathbf{b}\|_{\mathbb{V}_2} = |\operatorname{curl} \mathbf{b}|$ as well.

Let

$$\begin{aligned}\mathbb{V} &= \mathbb{V}_1 \times \mathbb{V}_2, \\ \mathbb{H} &= \mathbb{H}_1 \times \mathbb{H}_2.\end{aligned}$$

The space \mathbb{H} has the structure of a Hilbert space when equipped with the scalar product

$$(\Phi, \Psi) = (\mathbf{u}, \mathbf{v}) + (\mathbf{b}, \mathbf{C}), \quad (5)$$

for $\Phi = (\mathbf{u}; \mathbf{b}), \Psi = (\mathbf{v}; \mathbf{C}) \in \mathbb{H}$.

The space \mathbb{V} is a Banach space with norm

$$\|\Phi\|_{\mathbb{V}} = \|\mathbf{u}\|_2 + \|\mathbf{b}\|_1, \quad (6)$$

for $\Phi = (\mathbf{u}; \mathbf{b}) \in \mathbb{V}$.

For any Banach space X we denote by X^* its dual space and $\langle \phi, \mathbf{u} \rangle$ the value of $\phi \in X^*$ on $\mathbf{u} \in X$.

For any any $\mathbf{u} \in \mathbb{W}^{k+1,p}, k \geq 0$, we set

$$\mathcal{E}(\mathbf{u}) = \frac{1}{2} \left[(\nabla \mathbf{u}) + (\nabla \mathbf{u})^T \right].$$

Let us recall the following results whose proofs can be found in [28].

Lemma 2.1 (Korn's inequalities). *Let $1 < p < \infty$, $\beta = (\beta_1, \dots, \beta_l)$ be a multi-index such that $k = \sum_{i=1}^l \beta_i$, and $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Then there exists a positive constant $K(\Omega)$ such that*

$$K(\Omega) \|\mathbf{u}\|_{p,k+1} \leq \left(\int_{\Omega} \left| \frac{\partial^k \mathcal{E}(\mathbf{u})}{\partial x_{\beta_1} \dots \partial x_{\beta_l}} \right|^p dx \right)^{\frac{1}{p}}, \quad (7)$$

for any $\mathbf{u} \in \mathbb{W}^{k+1,p}$.

The following Poincaré's inequality is very crucial

$$\lambda_1 \|\mathbf{u}\|_2^2 \leq \|\mathbf{u}\|_1^2, \quad (8)$$

for any $\mathbf{u} \in \mathbb{H}^1(\Omega)$. This inequality holds with $\|\mathbf{u}\|_2$ for $\mathbf{u} \in \mathbb{H}^2(\Omega)$.

We also recall that there exist two positive constants $K_1(\Omega), K_2(\Omega)$ such that

$$K_1(\Omega) \|\mathbf{b}\|_{\mathbb{H}^2}^2 \leq |\Delta \mathbf{b}|^2 \leq K_2(\Omega) \|\mathbf{b}\|_{\mathbb{H}^2}^2, \quad (9)$$

for any $\mathbf{b} \in \mathbb{V}_2 \cap \mathbb{H}(\Omega)$.

There also holds

$$|\nabla \phi|^2 \leq |\phi|^2 + |\operatorname{curl} \phi|^2, \quad (10)$$

for any divergence free function ϕ satisfying $(\phi \cdot \mathbf{n})|_{\partial\Omega} = 0$. We refer to [22] or [25] for the proofs of (9) and (10).

3. EXISTENCE OF THE GLOBAL ATTRACTOR \mathcal{A}

In this section we will investigate the existence of a set $\mathcal{A} \subset \mathbb{H}$ which is the global attractor of $\mathbb{S}(t)$ in \mathbb{H} . But first of all we recall some definitions which are taken from [41].

Definition 3.1. A set $\mathcal{A} \subset \mathbb{H}$ is called an attractor for the semigroup $\mathbb{S}(t)$ if it enjoys the following properties:

- \mathcal{A} is invariant; that is, $\mathbb{S}(t)\mathcal{A} = \mathcal{A}$, for any $t \geq 0$,
- \mathcal{A} possesses an open neighborhood \mathcal{U} such that for any $\mathbf{u}_0 \in \mathcal{U}$

$$\lim_{t \rightarrow \infty} \inf_{y \in \mathcal{A}} |\mathbb{S}(t)\mathbf{u}_0 - y| = 0.$$

The set $\mathcal{A} \subset \mathbb{H}$ is said to be a global attractor if it is compact attractor and attracts any bounded sets of \mathbb{H} .

The notion of an absorbing set is very important for the investigation of the existence of a global attractor \mathcal{A} . We also recall the definition of an absorbing set \mathcal{B} .

Definition 3.2. Let \mathcal{B} be a bounded set of \mathbb{H} and \mathcal{U} be an open set containing \mathcal{B} . We say that \mathcal{B} is an absorbing in \mathcal{U} if for any bounded set $\mathcal{B}_0 \subset \mathcal{U}$, there exists a time $t_0 = t_0(\mathcal{B}_0)$ such that $\mathbb{S}(t)\mathcal{B}_0 \subset \mathcal{B}$, for any $t \geq t_0$.

After these definitions we formulate our first main result.

Theorem 3.3. *The semigroup $\mathbb{S}(t) : \mathbb{H} \rightarrow \mathbb{H}, t \geq 0$ has a global attractor \mathcal{A} which is maximal wrt to the inclusion in \mathbb{H} .*

To prove this theorem we start by finding absorbing sets in \mathbb{H} and \mathbb{V} .

Lemma 3.4. *There exists a ball $B_{\mathbb{H}}^{\rho_1} \subset \mathbb{H}$ which absorbs any bounded set of \mathbb{H} .*

Proof. The set \mathcal{V} is dense in \mathbb{H} , so it is permissible to take $\phi = \mathbf{u}$ in (3). From this operation we obtain that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} |\mathbf{u}(t)|^2 + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k} \right) + (\Gamma(\mathbf{u}(t)) \mathcal{E}(\mathbf{u}(t)), \mathcal{E}(\mathbf{u}(t))) \\ - \mu(\mathbf{b}(t) \cdot \nabla \mathbf{b}(t), \mathbf{u}(t)) - (f, \mathbf{u}(t)) = 0, \end{aligned} \quad (11)$$

where we have used the well-known fact that $(\mathbf{w} \cdot \nabla \mathbf{v}, \mathbf{v}) = 0$ for any $\mathbf{w} \in H^1(\Omega)$ and any divergence free function $\mathbf{v} \in H^1(\Omega)$. Using a similar reasoning we also have that

$$\frac{1}{2} \frac{d}{dt} |\mathbf{b}(t)|^2 - S(\Delta \mathbf{b}(t), \mathbf{b}(t)) - \mu(\mathbf{b}(t) \cdot \nabla \mathbf{u}(t), \mathbf{b}(t)) = 0. \quad (12)$$

Now summing up (11) and (12) side by side yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} (|\mathbf{u}(t)|^2 + |\mathbf{b}(t)|^2) + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k} \right) + (\Gamma(\mathbf{u}(t)) \mathcal{E}(\mathbf{u}(t)), \mathcal{E}(\mathbf{u}(t))) \\ - S(\Delta \mathbf{b}(t), \mathbf{b}(t)) - (f, \mathbf{u}(t)) = 0. \end{aligned} \quad (13)$$

Here the property $(\mathbf{b} \cdot \nabla \mathbf{b}, \mathbf{u}) = -(\mathbf{b} \cdot \nabla \mathbf{u}, \mathbf{b})$ was used. Since

$$\begin{aligned} (\Gamma(\mathbf{u}) \mathcal{E}(\mathbf{u}), \mathcal{E}(\mathbf{u})) &\geq 0, \\ \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u})}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u})}{\partial x_k} \right) &\geq \mu_1 K(\Omega) \|\mathbf{u}\|_2^2, \end{aligned}$$

and

$$-S(\Delta \mathbf{b}, \mathbf{b}) \geq S \|\mathbf{b}\|_1^2,$$

it follows from (13) that

$$\frac{1}{2} \frac{d}{dt} (|\mathbf{u}(t)|^2 + |\mathbf{b}(t)|^2) + \mu_1 K(\Omega) \|\mathbf{u}(t)\|_2^2 + S \|\mathbf{b}(t)\|_1^2 \leq |g| |\mathbf{u}(t)|. \quad (14)$$

Letting $\nu_0 = \min(\mu_1 K(\Omega), S)$ we can see from (14) along with Poincaré's inequality (8) and Young's inequality that

$$\frac{1}{2} \frac{d}{dt} (|\mathbf{u}(t)|^2 + |\mathbf{b}(t)|^2) + \nu_0 \lambda_1 (|\mathbf{u}(t)|^2 + |\mathbf{b}(t)|^2) \leq 2\delta^{-1} |g|^2 + \delta 2^{-1} (|\mathbf{u}(t)|^2 + |\mathbf{b}(t)|^2),$$

for any $\delta > 0$. Choosing $\delta = \nu_0$ we derive from the last estimate that

$$\frac{d}{dt} (|\mathbf{u}(t)|^2 + |\mathbf{b}(t)|^2) + \nu_0 \lambda_1 (|\mathbf{u}(t)|^2 + |\mathbf{b}(t)|^2) \leq \frac{4}{\nu_0} |g|^2. \quad (15)$$

Setting $\nu_1 = \frac{4}{\nu_0}$ and $y(t) = |\mathbf{u}(t)|^2 + |\mathbf{b}(t)|^2$ with $y(0) = |\mathbf{u}_0|^2 + |\mathbf{b}_0|^2$, the estimate (15) is equivalent to

$$\frac{dy(t)}{dt} + \nu_0 y(t) \leq \nu_1 |f|^2,$$

which together with Gronwall's lemma implies that

$$\begin{aligned} y(t) &\leq y(0) e^{-\nu_1 t} + \frac{\nu_1 |f|^2}{\nu_0} (1 - e^{-\nu_1 t}) \\ &\leq y(0) e^{-\nu_1 t} + \frac{\nu_1 |f|^2}{\nu_0}. \end{aligned}$$

Since $e^{-\nu_1 t} \rightarrow 0$ as $t \rightarrow \infty$, we can find a time $t_0 = t_0(\mathbf{u}_0, \mathbf{b}_0)$ such that

$$y(t) \leq \frac{2\nu_1 |f|^2}{\nu_0}, \quad (16)$$

for any $t \geq t_0$. This means that the ball in \mathbb{H}

$$B_{\mathbb{H}}^{\rho_1} = \{\mathbf{z} \in \mathbb{H}; |\mathbf{z}|^2 \leq \rho_1^2\},$$

with $\rho_1 > \left(\frac{2\nu_1|f|^2}{\nu_0}\right)^{1/2}$ is an absorbing set in \mathbb{H} . \square

Having found $B_h^{\rho_1}$ we prove that there is also a ball $B_{\mathbb{V}}^{\rho_2} \subset \mathbb{V}$ which attracts all bounded sets of \mathbb{H} . We postpone the proof for the next lemma. For now we establish an additional estimate that is going to play an important role in the aforementioned claim. From (14) we see that

$$\frac{dy(t)}{dt} + 2\nu_0 (\|\mathbf{u}(t)\|_2^2 + \|\mathbf{b}(t)\|_1^2) \leq 2|f|\|\mathbf{u}(t)\|, \quad (17)$$

for any $t \geq 0$. Integrating (17) over $[t, t+r]$, with $r > 0$ arbitrary, yields

$$y(t+r) + 2\nu_0 \int_t^{t+r} (\|\mathbf{u}(s)\|_2^2 + \|\mathbf{b}(s)\|_1^2) ds \leq y(t) + 2|f| \int_t^{t+r} \|\mathbf{u}(s)\| ds.$$

Whenever $t \geq t_0$, we have that

$$y(t+r) + 2\nu_0 \int_t^{t+r} (\|\mathbf{u}(s)\|_2^2 + \|\mathbf{b}(s)\|_1^2) ds \leq \rho_1^2 + 2|f|\rho_1 r. \quad (18)$$

Out of (18), we easily derive that

$$\int_t^{t+r} (\|\mathbf{u}(s)\|_2^2 + \|\mathbf{b}(s)\|_1^2) ds \leq \kappa_0(r), \quad (19)$$

where

$$\kappa_0(r) = \frac{1}{2\nu_0} (\rho_1^2 + 2|f|\rho_1 r).$$

We will now show that there is also a ball $B_{\mathbb{V}}^{\rho_2} \subset \mathbb{V}$ which absorbs all bounded set of \mathbb{H} .

Lemma 3.5. *The semigroup $\mathbb{S}(t)$ has a bounded absorbing set $B_{\mathbb{V}}^{\rho_2} \subset \mathbb{V}$.*

Proof. The proof of the result will consist of several steps.

Step 1: Uniform estimate wrt t of \mathbf{b} in \mathbb{V}_2

Taking $\psi = -\Delta \mathbf{b}(t)$ in (4) we obtain that

$$-\left(\frac{\partial \mathbf{b}(t)}{\partial t}, \Delta \mathbf{b}(t)\right) + S|\Delta \mathbf{b}(t)|^2 = \mu [(\mathbf{u}(t) \cdot \nabla \mathbf{b}(t), \Delta \mathbf{b}(t)) - (\mathbf{b}(t) \cdot \nabla \mathbf{u}(t), \Delta \mathbf{b}(t))],$$

which is equivalent to

$$\frac{1}{2} \frac{d}{dt} |\text{curl } \mathbf{b}(t)|^2 + S|\Delta \mathbf{b}(t)|^2 = \mu [(\mathbf{u}(t) \cdot \nabla \mathbf{b}(t), \Delta \mathbf{b}(t)) - (\mathbf{b}(t) \cdot \nabla \mathbf{u}(t), \Delta \mathbf{b}(t))]. \quad (20)$$

Since $|\text{curl } \mathbf{b}(t)|^2$ is the same as $\|\mathbf{b}(t)\|_1^2$ for $\mathbf{b}(t) \in \mathbb{V}_1$, we can rewrite (20) in the following form

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{b}(t)\|_1^2 + S|\Delta \mathbf{b}(t)|^2 = \mu [(\mathbf{u}(t) \cdot \nabla \mathbf{b}(t), \Delta \mathbf{b}(t)) - (\mathbf{b}(t) \cdot \nabla \mathbf{u}(t), \Delta \mathbf{b}(t))]. \quad (21)$$

Now we want to estimate the right hand side of (21). To do so we mainly use that fact that if $\mathbf{u} \in \mathbb{V}_1$, then $\mathbf{u} \in L^q(\Omega)$ for any $2 \leq q < \infty$ and $\nabla \mathbf{u} \in L^s(\Omega)$ with $2 \leq s \leq \frac{2n}{n-2}$ ($2 \leq s < \infty$ if $n = 2$.) For the first term we have

$$\begin{aligned} \mu(\mathbf{u} \cdot \nabla \mathbf{b}, \mathbf{b}) &\leq \mu c_1 \|\mathbf{u}\|_{L^\infty} \|\nabla \mathbf{b}\| |\Delta \mathbf{b}| \\ &\leq \delta^{-1} (\mu c_1)^2 \|\mathbf{u}\|_2^2 \|\nabla \mathbf{b}\|^2 + \delta |\Delta \mathbf{b}|^2. \end{aligned} \quad (22)$$

Owing to (10) and (22) we have that

$$\mu(\mathbf{u} \cdot \nabla \mathbf{b}, \Delta \mathbf{b}) \leq \delta^{-1}(\mu c_2)^2 \|\mathbf{u}\|_2^2 \|\mathbf{b}\|_1^2 + \delta |\Delta \mathbf{b}|^2. \quad (23)$$

By Hölder's inequality we see that the second term verifies

$$\begin{aligned} \mu(\mathbf{b} \cdot \nabla \mathbf{u}, \Delta \mathbf{b}) &\leq \mu c_3 \|\mathbf{b}\|_{L^4} \|\nabla \mathbf{u}\|_{L^4} |\Delta \mathbf{b}| \\ &\leq \delta^{-1}(\mu c_4)^2 \|\mathbf{b}\|_1^2 \|\mathbf{u}\|_2^2 + \delta |\Delta \mathbf{b}|^2. \end{aligned} \quad (24)$$

Let us set $\nu_1 = 2\delta^{-1}\mu \min(c_4, c_2)$. Owing to (23) and (24) we infer from (21) that

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{b}(t)\|_1^2 + S |\Delta \mathbf{b}(t)|^2 \leq \nu_2 \|\mathbf{b}(t)\|_1^2 \|\mathbf{u}(t)\|_2^2 + 2\delta |\Delta \mathbf{b}(t)|^2,$$

which with an appropriate choice of δ , let us say $\delta = \frac{S}{4}$, implies that

$$\frac{d}{dt} \|\mathbf{b}(t)\|_1^2 + S |\Delta \mathbf{b}(t)|^2 \leq \nu_1 \|\mathbf{b}(t)\|_1^2 \|\mathbf{u}(t)\|_2^2. \quad (25)$$

Because of (19) we can use the uniform Gronwall lemma (see, for instance, [41]) in (25) to get an uniform estimate wrt t of \mathbf{b} in \mathbb{V}_2 which does not explode too fast in time. This estimate reads

$$\|\mathbf{b}(t+r)\|_1^2 \leq \frac{\nu_1 \kappa_0(r)}{r} e^{\nu_1 \kappa_0(r)},$$

for any $r > 0$ and $t \geq t_0$. This implies that for any $r > 0$ there exists $\kappa_1(r) = \frac{\nu_1 \kappa_0(r)}{r} e^{\nu_1 \kappa_0(r)} > 0$ such that

$$\|\mathbf{b}(t)\|_1^2 \leq \kappa_1(r), \quad (26)$$

for any $t \geq t_0 + r$.

Step 2: Uniform estimate wrt t of $\Delta \mathbf{b}$ in $L^2(t, t+r; L^2(\Omega))$

For $t \geq t_0 + r$ we obtain from (25) and (26) that

$$\|\mathbf{b}(t+r)\|_1^2 + S \int_t^{t+r} |\Delta \mathbf{b}(s)|^2 ds \leq \nu_1 \kappa_1(r) \int_t^{t+r} \|\mathbf{u}(s)\|_2^2 ds + \|\mathbf{b}(t)\|_1^2,$$

which along with (19) and (26) enables us to deduce that

$$S \int_t^{t+r} |\Delta \mathbf{b}(s)|^2 ds \leq \kappa_1 (\nu_1 \kappa_0(r) + 1).$$

We deduce easily from this last estimate that for any $r > 0$

$$\int_t^{t+r} |\Delta \mathbf{b}(s)|^2 ds \leq \kappa_2(r), \quad (27)$$

for any $t \geq t_0 + r$. This inequality will be very useful for the next step.

Step 3: Uniform estimate wrt t of \mathbf{u} in \mathbb{V}_1

We want to establish an uniform estimate of $\partial \mathbf{u} / \partial t$ in \mathbb{H} . To shorten notation we will set $\mathbf{u}_t = \partial \mathbf{u} / \partial t$. We take $\phi = \mathbf{u}_t$ in (3) and we get

$$\begin{aligned} |\mathbf{u}_t(t)|^2 + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}_t(t))}{\partial x_k} \right) + (\Gamma(\mathbf{u}) \mathcal{E}(\mathbf{u}(t)), \mathcal{E}(\mathbf{u}_t)) + (\mathbf{u}(t) \cdot \nabla \mathbf{u}(t), \mathbf{u}_t(t)) \\ = \mu(\mathbf{b}(t) \cdot \nabla \mathbf{b}(t), \mathbf{u}_t(t)) + (f, \mathbf{u}_t(t)). \end{aligned} \quad (28)$$

Let Σ be the potential defined by

$$\Sigma(|\mathcal{E}|^2) = \int_0^{|\mathcal{E}|^2} \mu_0(\varepsilon + s)^{-\alpha/2} ds.$$

it is not difficult to see that

$$\frac{d}{dt} \Sigma(|\mathcal{E}(\mathbf{u}(t))|^2) = \mu_0(\varepsilon + |\mathcal{E}(\mathbf{u}(t))|)^{-\alpha/2} \mathcal{E}(\mathbf{u}(t)) \mathcal{E}(\mathbf{u}_t(t)),$$

and

$$\left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}_t(t))}{\partial x_k} \right) = \frac{d}{dt} \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k} \right)$$

Hence we deduce from (28) that

$$\begin{aligned} |\mathbf{u}_t(t)|^2 + \frac{d}{dt} \left(\int_{\Omega} \Sigma(\mathcal{E}(\mathbf{u}(t))) dx + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k} \right) \right) \\ \leq -(\mathbf{u}(t) \cdot \nabla \mathbf{u}(t), \mathbf{u}_t(t)) + \mu(\mathbf{b}(t) \cdot \nabla \mathbf{b}(t), \mathbf{u}_t(t)) + |f| |\mathbf{u}_t(t)|, \end{aligned} \quad (29)$$

which immediately implies that

$$\begin{aligned} \frac{1}{2} |\mathbf{u}_t(t)|^2 + \frac{d}{dt} \left(\int_{\Omega} \Sigma(\mathcal{E}(\mathbf{u}(t))) dx + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}_t(t))}{\partial x_k} \right) dx \right) \\ \leq -(\mathbf{u}(t) \cdot \nabla \mathbf{u}(t), \mathbf{u}_t(t)) + \mu(\mathbf{b}(t) \cdot \nabla \mathbf{b}(t), \mathbf{u}_t(t)) + \frac{1}{2} |f|^2, \\ \leq I_1 + I_2 + \frac{1}{2} |f|^2. \end{aligned} \quad (30)$$

For I_1 we have

$$\begin{aligned} |I_1| &\leq c_5 |\mathbf{u}(t)|_{L^\infty} |\nabla \mathbf{u}(t)| |\mathbf{u}_t(t)|, \\ &\leq c_6 \delta^{-1} \|\mathbf{u}(t)\|_2^2 |\nabla \mathbf{u}(t)|^2 + \delta |\mathbf{u}_t(t)|, \end{aligned} \quad (31)$$

for any $\delta > 0$. We can also check that

$$|I_2| \leq \delta^{-1} c_7 \|\mathbf{b}(t)\|_{L^\infty}^2 |\nabla \mathbf{b}(t)|^2 + \delta |\mathbf{u}_t(t)|^2,$$

for any $\delta > 0$. Thanks to (9) $\|\mathbf{b}\|_2^2$ is equivalent to $|\Delta \mathbf{b}|^2$ for any $\mathbf{b} \in \mathbb{V}_2 \cap \mathbb{V}_3$. Therefore, we can derive from the last estimate and (10) that

$$|I_2| \leq \delta^{-1} c_8 |\Delta \mathbf{b}(t)|^2 \|\mathbf{b}(t)\|_1^2 + \delta |\mathbf{u}_t(t)|^2. \quad (32)$$

By choosing $\delta = 1/8$, we can deduce from (30)-(32) that

$$\frac{1}{4} |\mathbf{u}_t(t)|^2 + \frac{dz(t)}{dt} \leq 8c_6 \|\mathbf{u}(t)\|_2^2 |\nabla \mathbf{u}|^2 + 8c_8 |\Delta \mathbf{b}(t)|^2 \|\mathbf{b}(t)\|_1^2, \quad (33)$$

where we have set

$$z(t) = \int_{\Omega} \Sigma(\mathcal{E}(\mathbf{u}(t))) dx + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k} \right).$$

Noticing that

$$\mu_1 K(\Omega) \|\mathbf{u}(t)\|_2^2 \leq \int_{\Omega} \Sigma(\mathcal{E}(\mathbf{u}(t))) dx + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}(t))}{\partial x_k} \right),$$

and dropping out the term $1/4 |\mathbf{u}_t(t)|^2$ we deduce from (33) that

$$\frac{dz(t)}{dt} \leq \frac{8c_6}{\mu_1} z(t) + 8c_8 |\Delta \mathbf{b}(t)|^2 \|\mathbf{b}(t)\|_1^2. \quad (34)$$

From here we want to use the Uniform Gronwall lemma, so we need to check that for certain t'_0 and any $r > 0$ there exist positive constants a_1, a_2, a_3 such that

$$\frac{8c_6}{\mu_1} \int_t^{t+r} |\nabla \mathbf{u}(s)|^2 ds \leq a_1, \quad (35)$$

$$8c_8 \int_t^{t+r} |\Delta \mathbf{b}(s)|^2 ds \|\mathbf{b}(s)\|_1^2 ds \leq a_2, \quad (36)$$

$$\int_t^{t+r} z(s) ds \leq a_3, \quad (37)$$

for any $t \geq t'_0$. Thanks to previous estimate we take $t'_0 = t_0 + r$ and infer from (19) that

$$\begin{aligned} \frac{8c_6}{\mu_1} \int_t^{t+r} |\nabla \mathbf{u}(s)|^2 ds &\leq \frac{8c_9}{\mu_1} \int_t^{t+r} \|\mathbf{u}(s)\|_2^2 ds \\ &\leq \frac{8c_9}{\mu_1} \kappa_0(r). \end{aligned}$$

Setting $a_1 = \frac{8c_9}{\mu_1} \kappa_0(r)$, we get (35). Invoking (26) and (27) we see that

$$\begin{aligned} 8c_8 \int_t^{t+r} |\Delta \mathbf{b}(s)|^2 \|\mathbf{b}(s)\|_1^2 ds &\leq 8c_8 \sup_{s \geq t_0+r} \|\mathbf{b}(s)\|_1^2 \int_t^{t+r} |\Delta \mathbf{b}(s)|^2 ds, \\ &\leq 8c_8 \kappa_1(r) \kappa_2(r). \end{aligned}$$

Letting $a_2 = 8c_8 \kappa_1(r) \kappa_2(r)$, we obtain (36). Now let us deal with (37). Thanks to (13) and (16) we have that

$$\begin{aligned} y(t+r) + 2\mu_1 \int_t^{t+r} \left(\frac{\partial \mathcal{E}(\mathbf{u}(s))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}(s))}{\partial x_k} \right) ds + 2 \int_t^{t+r} \left(\int_{\Omega} \Gamma(\mathbf{u}) |\mathcal{E}(\mathbf{u}(s))|^2 \right) ds \\ + 2S \int_t^{t+r} \|\mathbf{u}(s)\|_1^2 ds \leq 2\rho_1 |f| + y(t), \end{aligned} \quad (38)$$

for any $t \geq t_0 + r$. Keeping only the second term of the left hand side of (38) and using (16) we see that

$$\mu_1 \int_t^{t+r} \left(\frac{\partial \mathcal{E}(\mathbf{u}(s))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{u}(s))}{\partial x_k} \right) ds \leq \rho_1 (|f| + \rho_1). \quad (39)$$

To estimate the term involving $\Sigma(\mathcal{E}(\mathbf{u}))$. For this purpose we notice that the function $g(s) = \mu_0(\varepsilon + s)^{-\alpha/2}$ is decreasing for $s \in [0, \infty)$. Hence

$$\Sigma(\mathcal{E}(\mathbf{u})) \leq \int_0^{|\mathcal{E}(\mathbf{u})|^2} (\sup_{s \geq 0} g(s)) ds = \frac{\mu_0}{\varepsilon^{\alpha/2}} |\mathcal{E}(\mathbf{u})|^2.$$

This implies that for any $t \geq t_0 + 2$

$$\int_t^{t+r} \left(\int_{\Omega} \Sigma(\mathcal{E}(\mathbf{u}(s))) \right) ds \leq \frac{\mu_0}{\varepsilon^{\alpha/2}} \int_t^{t+r} |\mathcal{E}(\mathbf{u}(s))|^2 ds.$$

Invoking korn's inequality and (19) we have that

$$\int_t^{t+r} \left(\int_{\Omega} \Sigma(\mathcal{E}(\mathbf{u}(s))) \right) ds \leq \frac{\mu_0}{\varepsilon^{\alpha/2}} K(\Omega) \kappa_0(r) r.$$

So putting $a_3 = \rho_1(|f| + \rho_1) + \frac{\mu_0}{\varepsilon^{\alpha/2}} K(\Omega) \kappa_0(r)r$, we get (37). Now we can apply the Uniform Gronwall lemma to (34) and we get

$$z(t+r) \leq \left(\frac{a_3}{r} + a_2 \right) \exp(a_1), \quad (40)$$

for any $t \geq t_0 + r$. Korn's inequality along with (40) implies that for any

$$\|\mathbf{u}(t)\|_2^2 \leq \frac{1}{\mu_1 K(\Omega)} \left(\frac{a_3}{r} + a_2 \right) \exp(a_1), \quad (41)$$

for any $t \geq t_0 + 2r$.

Let us set

$$\kappa_3(r) = \kappa_1(r) + \frac{1}{\mu_1 K(\Omega)} \left(\frac{a_3}{r} + a_2 \right) \exp(a_1).$$

So combining (26) and (41) we obtain that

$$\|\mathbf{u}(t)\|_2^2 + \|\mathbf{b}(t)\|_1^2 \leq \kappa_3(r), \quad (42)$$

for any $t \geq t_0 + 2r$. The equation (42) implies that the set

$$B_V^{\rho_2} = \{\mathbf{z} \in \mathbb{V} : \|\mathbf{z}\|_V^2 \leq \rho_2^2\},$$

with $\rho_2 = \sqrt{\kappa_3(r)}$, is an attracting set in $\mathbb{V} = \mathbb{V}_1 \times \mathbb{V}_2$. Here we have set

$$\|\mathbf{z}\|_V^2 = \|\mathbf{u}\|_2^2 + \|\mathbf{b}\|_1^2,$$

for any $\mathbf{z} = (\mathbf{u}; \mathbf{b}) \in \mathbb{V}$. □

Proof of Theorem 3.3. It is now standard to prove the existence of the global attractor. The set $B_{\mathbb{H}}^{\rho_1}$ is absorbing in \mathbb{H} and there exists t_0 such that $\mathbb{S}(t)B_{\mathbb{H}}^{\rho_1} \subset B_V^{\rho_2}$, for any $t \geq t_0$. Moreover, $B_V^{\rho_2}$ is compact in \mathbb{H} , hence $\cup_{t \geq t_0} \mathbb{S}(t)B_{\mathbb{H}}^{\rho_1}$ is compact and we can deduce from [41, Theorem I.1.1] that $\mathcal{A} = \omega(B_{\mathbb{H}}^{\rho_1}) = \overline{\cap_{t \geq 0} \cup_{t \geq t_0} \mathbb{S}(t)B_{\mathbb{H}}^{\rho_1}}$ is a compact attractor which is maximal wrt the inclusion in \mathbb{H} . This completes the proof of the theorem. □

Our next concern is to find an estimate of the bounds for $d_H(\mathcal{A})$ and $d_f(\mathcal{A})$, the Hausdorff and fractal dimension of \mathcal{A} . This procedure needs that $\mathbb{S}(t)$ is Fréchet differentiable wrt to the initial data. Will show this fact in the next section.

4. UNIFORM DIFFERENTIABILITY OF $\mathbb{S}(t)$

Our second result is about the regularity of the semigroup $\{\mathbb{S}(t) : t \geq 0\}$ with respect to the initial data of the problem (2). We recall the following definition which is borrowed from [33].

Definition 4.1. We say that $\mathbb{S}(t)$ is uniformly differentiable on \mathcal{A} if for every $(\mathbf{u}; \mathbf{b}) \in \mathcal{A}$ there exists a linear operator $\mathfrak{L}(\mathbf{u}; \mathbf{b})$, such that, for all $t \geq 0$;

$$\sup_{\{(\mathbf{v}; \mathbf{c}), (\mathbf{u}; \mathbf{b}) \in \mathcal{A} : |\mathbf{v} - \mathbf{u}| + |\mathbf{c} - \mathbf{b}| < \varkappa\}} \frac{|\mathbb{S}(t)(\mathbf{v}; \mathbf{c}) - \mathbb{S}(t)(\mathbf{u}; \mathbf{b}) - \mathfrak{L}(\mathbf{u}; \mathbf{b})(\mathbf{v} - \mathbf{u}; \mathbf{c} - \mathbf{b})|}{|\mathbf{v} - \mathbf{u}| + |\mathbf{c} - \mathbf{b}|} \rightarrow 0,$$

as $\varkappa \rightarrow 0$, and

$$\sup_{(\mathbf{u}; \mathbf{b}) \in \mathcal{A}} \sup_{(\mathbf{v}; \mathbf{c}) : |\mathbf{v}| + |\mathbf{c}| \neq 0} \frac{|\mathfrak{L}(\mathbf{u}; \mathbf{b})(\mathbf{v}; \mathbf{c})|}{|\mathbf{v}| + |\mathbf{c}|} < \infty, \text{ for each } t \geq 0.$$

The main result of this section is stated in the following

Theorem 4.2. *The semigroup $\mathbb{S}(t), t \geq 0$ is uniformly differentiable on \mathcal{A} .*

Proof. We start the proof by showing that $\mathbb{S}(t)$ is Lipschitz continuous. For doing so we consider two weak solutions $(\mathbf{u}; \mathbf{b})$ and $(\mathbf{v}; \mathbf{c})$ associated to the same forcing f and the initial data $(\mathbf{u}_0; \mathbf{b}_0)$ and $(\mathbf{v}_0; \mathbf{c}_0)$, respectively. The functions $\mathbf{w} = \mathbf{v} - \mathbf{u}$ and $\mathbf{m} = \mathbf{c} - \mathbf{b}$ satisfy

$$\begin{aligned} & (\mathbf{w}_t(t), \phi) + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{w}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\phi)}{\partial x_k} \right) + \left(\Gamma(\mathbf{v}(t)) \mathcal{E}(\mathbf{v}(t)) - \Gamma(\mathbf{u}(t)) \mathcal{E}(\mathbf{u}(t)), \mathcal{E}(\phi) \right) \\ & + (\mathbf{v}(t) \cdot \nabla \mathbf{w}(t) + \mathbf{w}(t) \cdot \nabla \mathbf{u}(t), \phi) + \mu \left[(\mathbf{b}(t) \cdot \nabla \phi, \mathbf{m}(t)) + (\mathbf{m}(t) \cdot \nabla \phi, \mathbf{c}(t)) \right] = 0, \end{aligned} \quad (43)$$

and

$$(\mathbf{m}_t, \psi) - S(\Delta, \psi) + \mu \left((\mathbf{u}(t) \cdot \nabla \mathbf{m}(t) - \mathbf{m}(t) \cdot \nabla \mathbf{u}(t) - \mathbf{c}(t) \cdot \nabla \mathbf{w}(t) + \mathbf{w}(t) \cdot \nabla \mathbf{c}(t), \psi) \right) = 0, \quad (44)$$

where we have set $\mathbf{w}_t = \frac{\partial \mathbf{w}}{\partial t}$ and $\mathbf{m}_t = \frac{\partial \mathbf{m}}{\partial t}$. Taking $\phi = \mathbf{w}$ and $\psi = \mathbf{m}$ in the above equations yields

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |\mathbf{w}(t)|^2 + \mu_1 \left(\frac{\partial \mathcal{E}(\mathbf{w}(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\mathbf{w}(t))}{\partial x_k} \right) + \left(\Gamma(\mathbf{v}(t)) \mathcal{E}(\mathbf{v}(t)) - \Gamma(\mathbf{u}(t)) \mathcal{E}(\mathbf{u}(t)), \mathcal{E}(\mathbf{w}(t)) \right) \\ & - (\mathbf{w}(t) \cdot \nabla \mathbf{w}(t), \mathbf{u}(t)) + \mu \left[(\mathbf{b}(t) \cdot \nabla \mathbf{w}(t), \mathbf{m}(t)) + (\mathbf{m}(t) \cdot \nabla \mathbf{w}(t), \mathbf{c}(t)) \right] = 0 \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} |\mathbf{m}(t)|^2 + S \|\mathbf{b}\|_1^2 = \mu \left((\mathbf{m}(t) \cdot \nabla \mathbf{u}(t), \mathbf{m}(t)) + (\mathbf{b}(t) \cdot \nabla \mathbf{w}(t), \mathbf{m}(t)) \right. \\ & \left. + (\mathbf{w}(t) \cdot \nabla \mathbf{m}(t), \mathbf{c}(t)) \right). \end{aligned}$$

Let $\Phi(t) = |\mathbf{w}(t)|^2 + |\mathbf{m}(t)|^2$. Summing up side by side and applying Korn's inequality we obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \Phi(t) + \nu_0 (|\mathbf{w}(t)|_2^2 + |\mathbf{m}(t)|_2^2) \leq |(\mathbf{w}(t) \cdot \nabla \mathbf{w}(t), \mathbf{u}(t))| + \mu \left| (\mathbf{m}(t) \cdot \nabla \mathbf{w}(t), \mathbf{c}(t)) \right| \\ & + \mu \left| (\mathbf{m}(t) \cdot \nabla \mathbf{u}(t), \mathbf{m}(t)) + (\mathbf{w}(t) \cdot \nabla \mathbf{m}(t), \mathbf{c}(t)) \right| \end{aligned} \quad (45)$$

To estimate the right hand side of (45) we will mainly use the fact that if $(\mathbf{u}_0; \mathbf{b}_0)$ and $(\mathbf{v}_0; \mathbf{c}_0)$ are an element of \mathcal{A} then $(\mathbf{u}(t); \mathbf{b}(t))$ and $(\mathbf{v}(t); \mathbf{c}(t))$ are element of $B_V^{\rho_2}$ for any $t \geq 0$. For the first term we have that

$$|(\mathbf{w}(t) \cdot \nabla \mathbf{w}(t), \mathbf{u}(t))| \leq c_1 \|\mathbf{u}(t)\| \|\mathbf{w}(t)\| |\nabla \mathbf{w}(t)|,$$

where we have used Hölder's inequality and the embedding $H^2(\Omega) \subset L^\infty(\Omega)$. By a similar argument we have

$$\begin{aligned} & |(\mathbf{m}(t) \cdot \nabla, \mathbf{c}(t)) + (\mathbf{w}(t) \cdot \nabla \mathbf{m}(t), \mathbf{c}(t))| \leq c_3 |\mathbf{m}(t)| |\nabla \mathbf{w}(t)|_{L^4} |\mathbf{c}(t)|_{L^4} \\ & + c_1 |\mathbf{m}(t)| |\nabla \mathbf{c}(t)| \|\mathbf{w}(t)\|_{L^\infty}, \\ & \leq \tilde{c}_3 |\mathbf{m}(t)| \|\mathbf{w}(t)\|_2 \|\mathbf{c}(t)\|_1. \end{aligned}$$

We can also check that

$$\begin{aligned} |(\mathbf{m}(t) \cdot \nabla \mathbf{u}(t), \mathbf{m}(t))| &\leq c_3 |\mathbf{m}(t)| |\nabla \mathbf{u}(t)|_{L^4} |\mathbf{m}(t)|_{L^4} \\ &\leq \bar{c}_3 |\mathbf{m}(t)| \|\mathbf{m}(t)\|_1 \|\mathbf{u}(t)\|_2. \end{aligned}$$

Since $(\mathbf{u}(t), \mathbf{b}(t))$ and $(\mathbf{v}(t), \mathbf{c}(t))$ are element of $B_V^{\rho_2}$, we can find a positive constant c_{10} such that the right hand side of (45) is bounded from above by

$$2\rho_2 c_{10} (\|\mathbf{w}(t)\| \|\mathbf{w}(t)\|_2 + |\mathbf{m}(t)| \|\mathbf{m}(t)\|_1 + |\mathbf{m}(t)| \|\Delta \mathbf{m}(t)\|),$$

where we used the fact that $\|\mathbf{m}\|_2$ and $|\Delta \mathbf{m}|$ for $\mathbf{m} \in \mathbb{V}_2 \cap H^2(\Omega)$. Hence

$$\frac{1}{2} \frac{d}{dt} \Phi(t) + \nu_0 (\|\mathbf{w}(t)\|_2^2 + \|\mathbf{m}(t)\|_2^2) \leq 4\rho_2 c_{10} \left((|\mathbf{m}(t)| + |\mathbf{w}(t)|) (\|\mathbf{w}(t)\|_2 + \|\mathbf{m}(t)\|_1) \right),$$

from which we can infer that

$$\frac{d}{dt} \Phi(t) + 2\nu_0 (\|\mathbf{w}(t)\|_2^2 + \|\mathbf{m}(t)\|_2^2) \leq 2\kappa^{-1} (8\rho_2 c_{10})^2 \Phi(t) + \kappa (\|\mathbf{w}(t)\|_2^2 + \|\mathbf{m}(t)\|_1^2),$$

for any $\kappa > 0$. By choosing $\kappa = \nu_0$, we deduce from the last inequality that

$$\frac{d}{dt} \Phi(t) + \nu_0 (\|\mathbf{w}(t)\|_2^2 + \|\mathbf{m}(t)\|_2^2) \leq 2\kappa^{-1} (8\rho_2 c_{10})^2 \Phi(t).$$

We put $\eta_1 = 2\kappa^{-1} (8\rho_2 c_{10})^2$ and use Gronwall's lemma to deduce that

$$\Phi(t) \leq (|\mathbf{w}_0|^2 + |\mathbf{m}_0|^2) \exp(\eta_1 t), \quad (46)$$

for any $t \geq 0$. This show that $\mathbb{S}(t)$ is Lipschitz continuous wrt to the initial data, which show also the uniqueness of the solution of (2).

Let $(\mathbf{u}; \mathbf{b})$ a weak solutions of (2). Using the same argument as in [5] it can be shown that the linearization of (2) about $(\mathbf{u}; \mathbf{b})$ is given by the following system of linear PDEs:

$$\begin{aligned} \left(\frac{\partial \mathbf{U}(t)}{\partial t}, \phi \right) + (\Gamma(\mathbf{u}(t)) \mathcal{E}(\mathbf{u}(t)) - \alpha \mathbf{A}_{ijkl}(\mathbf{u}(t)) \mathcal{E}(\mathbf{u}(t)), \mathcal{E}(\phi)) + \mu_1 (\mathcal{E}(\mathbf{u}(t)), \mathcal{E}(\phi)) \\ = \mu \left((\mathbf{M}(t) \cdot \nabla \mathbf{b}(t), \phi) + (\mathbf{b}(t) \cdot \nabla \mathbf{M}(t), \phi) \right) + (\mathbf{U}(t) \cdot \nabla \phi, \mathbf{u}(t)) \\ + (\mathbf{u}(t) \cdot \nabla \phi, \mathbf{U}(t)), \quad (47) \end{aligned}$$

$$\begin{aligned} \left(\frac{\partial \mathbf{M}(t)}{\partial t}, \psi \right) - S(\Delta \mathbf{M}(t), \psi) = -\mu \left((\mathbf{U}(t) \cdot \nabla \mathbf{b}(t), \psi) + (\mathbf{u}(t) \cdot \nabla \mathbf{M}(t), \psi) \right. \\ \left. - (\mathbf{M}(t) \cdot \nabla \mathbf{u}(t), \psi) - (\mathbf{b}(t) \cdot \nabla \mathbf{U}(t), \psi) \right). \quad (48) \end{aligned}$$

Here the tensor \mathbf{A}_{ijkl} is defined by

$$\mathbf{A}_{ijkl}(\mathbf{u}) = \mu_0 (\varepsilon + |\mathcal{E}(\mathbf{u})|^2)^{1+\alpha/2} \mathcal{E}_{ij}(\mathbf{u}) \mathcal{E}_{kl}(\mathbf{u}), \quad (49)$$

and it satisfies

$$\begin{aligned} \int_{\Omega} \left(\Gamma(\mathbf{u}) |\mathcal{E}(\mathbf{U})|^2 - \alpha \mathbf{A}_{ijkl}(\mathbf{u}) \mathcal{E}_{ij}(\mathbf{U}) \mathcal{E}_{kl}(\mathbf{U}) \right) dx \\ \geq 2\varepsilon \alpha \mu_0 \int_{\Omega} \frac{|\mathcal{E}(\mathbf{U})|^2}{(\varepsilon + |\mathcal{E}(\mathbf{u})|^2)^{1+\alpha/2}} dx + 2(1-\alpha) \mu_0 \int_{\Omega} \frac{\mathcal{E}_{ij}(\mathbf{U}) \mathcal{E}_{kl}(\mathbf{U}) \mathcal{E}_{ij}(\mathbf{u}) \mathcal{E}_{kl}(\mathbf{u})}{(\varepsilon + |\mathcal{E}(\mathbf{u})|^2)^{\alpha/2}}, \quad (50) \end{aligned}$$

for all $\varepsilon, \mu_0 \geq 0$, $0 \leq \alpha < 1$. Note that by standard Galerkin method we can show that (47)-(48) has a unique solution $(\mathbf{U}; \mathbf{M}) \in L_{loc}^\infty(0, \infty; \mathbb{H}) \cap L_{loc}^2(0, \infty; \mathbb{V})$ which implies the second condition in Definition 4.1. Now let $\Theta = \mathbf{v} - \mathbf{u} - \mathbf{U}$ and $\Psi = \mathbf{b} - \mathbf{c} - \mathbf{M}$. After some algebra

$$\begin{aligned} & \left(\frac{\partial \Theta(t)}{\partial t}, \phi \right) + \mu_1 \left(\frac{\partial \mathcal{E}(\Theta(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\phi)}{\partial x_k} \right) - \left(\Gamma(\mathbf{u}(t)) \mathcal{E}(\mathbf{U}(t)), \mathcal{E}(\phi) \right) \\ & + \alpha (\mathbf{A}_{ijkl}(\mathbf{u}(t)) \mathcal{E}(\mathbf{U}(t)), \mathcal{E}(\phi)) + \mu \left((\mathbf{b}(t) \cdot \nabla \phi, \Psi(t)) + (\Psi(t) \cdot \nabla \phi, \mathbf{b}(t)) \right) \\ & + (\mathbf{v}(t) \cdot \nabla \Theta(t) + \Theta(t) \cdot \nabla \mathbf{v}(t), \phi) - (\mathbf{w}(t) \cdot \nabla \mathbf{w}(t), \phi) + (\mathbf{m}(t) \cdot \nabla \mathbf{m}(t), \phi) = 0, \end{aligned} \quad (51)$$

and

$$\begin{aligned} & \left(\frac{\partial \Psi(t)}{\partial t}, \psi \right) - S(\Delta \Psi(t), \psi) + \mu \left(\Theta(t) \cdot \nabla \mathbf{c}(t) - \mathbf{b}(t) \cdot \nabla \Theta(t) + \mathbf{v}(t) \cdot \nabla \Psi(t) \right. \\ & \left. - \Psi(t) \cdot \nabla \mathbf{v}(t) + \mathbf{m}(t) \cdot \nabla \mathbf{w}(t) - \mathbf{U}(t) \cdot \nabla \mathbf{m}(t), \psi \right) = 0. \end{aligned} \quad (52)$$

Taking $\phi = \Theta$ (resp., $\psi = \Psi$) in the (51) (resp., (52)) yields

$$\begin{aligned} & \frac{1}{2} \frac{d|\Theta(t)|^2}{dt} + \mu_1 \left(\frac{\partial \mathcal{E}(\Theta(t))}{\partial x_k}, \frac{\partial \mathcal{E}(\Theta(t))}{\partial x_k} \right) + \left(\Gamma(\mathbf{v}(t)) \mathcal{E}(\mathbf{v}(t)) - \Gamma(\mathbf{u}(t)) \mathcal{E}(\mathbf{u}(t)), \Theta(t) \right) \\ & - \left(\Gamma(\mathbf{u}(t)) \mathcal{E}(\mathbf{U}(t)), \mathcal{E}(\Theta(t)) \right) + \alpha (\mathbf{A}_{ijkl}(\mathbf{u}(t)) \mathcal{E}(\mathbf{U}(t)), \mathcal{E}(\Theta(t))) + (\Theta \cdot \nabla \mathbf{v}(t), \Theta(t)) \\ & = (\mathbf{w}(t) \cdot \nabla \mathbf{w}(t), \Theta(t)) - \mu \left((\mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t)) + (\Psi(t) \cdot \nabla \Theta(t), \mathbf{b}(t)) \right. \\ & \left. + (\mathbf{m}(t) \cdot \nabla \mathbf{m}(t), \Theta(t)) \right), \end{aligned}$$

and

$$\begin{aligned} & \frac{1}{2} \frac{d|\Psi(t)|^2}{dt} + S\|\Psi(t)\|_1^2 + \mu \left((\Theta(t) \cdot \nabla \mathbf{c}(t) - \mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t)) - (\Psi(t) \cdot \nabla \mathbf{v}(t), \Psi(t)) \right. \\ & \left. + (\mathbf{m}(t) \cdot \nabla \mathbf{w}(t) - \mathbf{U}(t) \cdot \nabla \mathbf{m}(t), \Psi(t)) \right) = 0 \end{aligned} \quad (53)$$

By using some results from [5, Page 150-151] we see that

$$\begin{aligned} & \frac{1}{2} \frac{d|\Theta(t)|^2}{dt} + \mu_1 K(\Omega) \|\Theta(t)\|_2^2 + (\Theta(t) \cdot \nabla \mathbf{v}(t), \Theta(t)) - (\mathbf{w}(t) \cdot \nabla \mathbf{w}(t), \Theta(t)) \\ & \leq -\mu \left((\mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t)) + (\Psi(t) \cdot \nabla \Theta(t), \mathbf{b}(t)) + (\mathbf{m}(t) \cdot \nabla \mathbf{m}(t), \Theta(t)) \right) \\ & + \int_{\Omega} \Sigma_{ijklmn} \mathcal{E}_{ij}(\Theta(t)) \mathcal{E}_{kl}(\mathbf{w}(t)) \mathcal{E}_{mn}(\mathbf{w}(t)) dx, \end{aligned} \quad (54)$$

where

$$\Sigma_{ijklmn} = \int_0^1 \int_0^1 \frac{\partial^3 \Sigma}{\partial \mathcal{E}_{ij} \partial \mathcal{E}_{kl} \partial \mathcal{E}_{mn}} (\mathcal{E}(\mathbf{u}(t) + \sigma \tau \mathbf{w}(t))) \tau d\tau d\sigma \quad (55)$$

We also have

$$\begin{aligned} \frac{1}{2} \frac{d|\Psi(t)|^2}{dt} + S\|\Psi(t)\|_1^2 \leq \mu \left| (\Theta(t) \cdot \nabla \mathbf{c}(t) - \mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t)) - (\Psi(t) \cdot \nabla \mathbf{v}(t), \Psi(t)) \right. \\ \left. + (\mathbf{m}(t) \cdot \nabla \mathbf{w}(t) - \mathbf{U}(t) \cdot \nabla \mathbf{m}(t), \Psi(t)) \right| \end{aligned} \quad (56)$$

Now we want to estimate each term in the right hand side of (54). We have that

$$\begin{aligned} \left(\Theta(t) \cdot \nabla \mathbf{u}(t), \Theta(t) \right) &\leq c_1 |\Theta(t)| \|\nabla \Theta(t)\| |\mathbf{u}(t)|_{L^\infty}, \\ &\leq C(\Omega) |\Theta(t)| \|\Theta(t)\|_2 \|\mathbf{u}(t)\|_2. \end{aligned}$$

Owing to Young's inequality and the fact that $\mathbf{u}(t) \in B_V^{\rho_2}$ for any $t \geq 0$ and , we obtain that

$$\left(\Theta(t) \cdot \nabla \mathbf{u}(t), \Theta(t) \right) \leq C(\Omega)^2 \rho_2^2 \varkappa^{-1} |\Theta(t)|^2 + \varkappa \|\Theta(t)\|_2^2, \quad (57)$$

for any $\varkappa > 0$.

Hölder's inequality implies that

$$\left| \left(\mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t) \right) + \left(\Psi(t) \cdot \nabla \Theta(t), \mathbf{b}(t) \right) \right| \leq 2C(\Omega) |\mathbf{b}(t)|_{L^4} |\nabla \Theta(t)|_{L^4} |\Psi(t)|. \quad (58)$$

Using fact that $H^1(\Omega) \subset L^4(\Omega)$ for $n = 2, 3$, and the $\|\psi\|_{H^1}$ and $|\text{curl } \psi|$ for any $\psi \in \mathbb{V}_2$ we deduce from the last inequality that

$$\left| \left(\mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t) \right) + \left(\Psi(t) \cdot \nabla \Theta(t), \mathbf{b}(t) \right) \right| \leq 2C(\Omega) \|\mathbf{b}(t)\|_1 \|\Theta(t)\|_2 |\Psi(t)|. \quad (59)$$

Since $(\mathbf{u}; \mathbf{b}) \in B_V^{\rho_2}$, we easily see that $\|\mathbf{b}(t)\|_1 \leq \rho_2$. Therefore

$$\mu \left| \left(\mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t) \right) + \left(\Psi(t) \cdot \nabla \Theta(t), \mathbf{b}(t) \right) \right| \leq \varkappa^{-1} (2\mu C(\Omega) \rho_2)^2 |\Psi(t)|^2 + \varkappa \|\Theta(t)\|_2^2. \quad (60)$$

It is not difficult to show that

$$\mu \left| \left(\mathbf{b}(t) \cdot \nabla \mathbf{m}(t), \Theta(t) \right) \right| \leq \mu C(\Omega) \|\mathbf{m}(t)\|_1 \|\Theta(t)\|_2 |\mathbf{m}(t)|.$$

Since $(\mathbf{w}; \mathbf{m}) \in B_V^{\rho_2}$, we have $\|\mathbf{m}\|_1 \leq \rho_2$. Thus,

$$\mu \left| \left(\mathbf{b}(t) \cdot \nabla \mathbf{m}(t), \Theta(t) \right) \right| \leq \varkappa^{-1} (\mu \rho_2 C(\Omega))^2 |\mathbf{m}(t)|^2 + \varkappa \|\Theta(t)\|_2^2. \quad (61)$$

We also check that

$$\begin{aligned} \left| \left(\mathbf{w}(t) \cdot \nabla \mathbf{w}(t), \Theta(t) \right) \right| &\leq C(\Omega) |\mathbf{w}(t)| \|\Theta(t)\|_2 \|\mathbf{w}(t)\|_2, \\ &\leq \varkappa^{-1} (C(\Omega) \rho_2)^2 |\mathbf{w}(t)|_2^2 + \varkappa \|\Theta(t)\|_2^2. \end{aligned} \quad (62)$$

As far as the term involving Σ_{ijklmn} is concerned we have that

$$\left| \int_{\Omega} \Sigma_{ijklmn} \mathcal{E}_{ij}(\Theta(t)) \mathcal{E}_{kl}(\mathbf{w}(t)) \mathcal{E}_{mn}(\mathbf{w}(t)) dx \right| \leq \varkappa^{-1} (C(\Omega) \rho_2)^2 \|\mathbf{w}(t)\|_2^2 + \varkappa \|\Theta(t)\|_2^2, \quad (63)$$

where we have used again the fact that $\|\mathbf{w}(t)\|_2 \leq \rho_2$. Putting (57)-(63) in (54) yields

$$\begin{aligned} \frac{1}{2} \frac{d|\Theta(t)|^2}{dt} + \mu_1 K(\Omega) \|\Theta(t)\|_2^2 &\leq \varkappa^{-1} (C(\Omega) \rho_2)^2 \left(|\Theta(t)|^2 + 4\mu^2 |\Psi(t)|^2 + \mu^2 |\mathbf{m}(t)|^2 \right. \\ &\quad \left. + |\mathbf{w}(t)|^2 + \|\mathbf{w}(t)\|_2^2 \right) + 5\varkappa \|\Theta(t)\|_2^2 \end{aligned} \quad (64)$$

In the next few lines we estimate each term in the right hand side of (56). Owing to the same argument we used for (59) and the fact that $|\nabla \mathbf{b}| \leq |\operatorname{curl} \mathbf{b}| + |\mathbf{b}|$ we obtain that

$$\begin{aligned} \left| \left(\Theta(t) \cdot \nabla \mathbf{c}(t) - \mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t) \right) \right| &\leq C(\Omega) \left(|\nabla \mathbf{c}(t)| \|\Theta(t)\|_2 |\Psi(t)| \right. \\ &\quad \left. + \|\mathbf{b}(t)\|_1 \|\Theta(t)\|_2 |\Psi(t)| \right), \\ &\leq C(\Omega) \left((\|\mathbf{c}(t)\|_1 + |\mathbf{c}(t)|) \|\Theta(t)\|_2 |\Psi(t)| \right. \\ &\quad \left. + \|\mathbf{b}(t)\|_1 \|\Theta(t)\|_2 |\Psi(t)| \right). \end{aligned}$$

Since $(\mathbf{u}; \mathbf{b})$ and $(\mathbf{v}; \mathbf{c})$ belong to $B_{\mathbb{V}}^{\rho_2}$ we have that

$$\begin{aligned} \left| \left(\Theta(t) \cdot \nabla \mathbf{c}(t) - \mathbf{b}(t) \cdot \nabla \Theta(t), \Psi(t) \right) \right| &\leq 3\mu\rho_2 C(\Omega) |\Psi(t)| \|\Theta(t)\|_2 \\ &\leq \varkappa^{-1} (3\mu\rho_2 C(\Omega))^2 |\Psi(t)|^2 + \varkappa \|\Theta(t)\|_2^2. \end{aligned} \quad (65)$$

Using the similar idea as used for (59) and (65) we see that

$$\begin{aligned} \mu \left| \left(\Psi(t) \cdot \nabla \mathbf{u}(t), \Psi(t) \right) \right| &\leq \mu C(\Omega) |\Psi(t)|_{L^4} |\nabla \mathbf{u}(t)|_{L^4} |\Psi(t)|, \\ &\leq \mu C(\Omega) \|\Psi(t)\|_1 \|\mathbf{u}(t)\|_2 |\Psi(t)|, \\ &\leq \gamma^{-1} (\mu\rho_2 C(\Omega))^2 |\Psi(t)|^2 + \gamma \|\Psi(t)\|_1^2. \end{aligned} \quad (66)$$

Before we proceed further we recall that $\mathbf{U} = \mathbf{w} - \Theta$. Hence

$$\left(\mathbf{m}(t) \cdot \nabla \mathbf{w}(t) - \mathbf{U}(t) \cdot \nabla \mathbf{m}(t), \Psi(t) \right) = \left(\mathbf{m}(t) \cdot \nabla \mathbf{w}(t) - \mathbf{w}(t) \cdot \nabla \mathbf{m}(t) + \Theta(t) \cdot \nabla \mathbf{m}(t), \Psi(t) \right).$$

As for (66) we can check that

$$\mu \left| \left(\mathbf{m}(t) \cdot \nabla \mathbf{w}(t), \Psi(t) \right) \right| \leq \gamma^{-1} (\mu\rho_2 C(\Omega))^2 |\mathbf{m}(t)|^2 + \gamma \|\Psi(t)\|_1^2. \quad (67)$$

We also have that

$$\begin{aligned} \left| \left(\mathbf{w}(t) \cdot \nabla \mathbf{m}(t), \Psi(t) \right) \right| &\leq C(\Omega) |\mathbf{w}(t)|_{L^\infty} |\nabla \Psi(t)| |\mathbf{m}(t)|, \\ &\leq C(\Omega) \left(\|\mathbf{w}(t)\|_2 \|\Psi(t)\|_1 |\mathbf{m}(t)| + \|\mathbf{w}(t)\|_2 |\mathbf{m}(t)| \|\Psi(t)\| \right). \end{aligned}$$

If $(\mathbf{u}_0; \mathbf{b}_0) \in \mathcal{A}$ and $(\mathbf{v}_0; \mathbf{c}_0) \in \mathcal{A}$, then $(\mathbf{w}(t); \mathbf{m}(t)) \in B_{\mathbb{V}}^{\rho_2}$. Hence we derive from the last estimate that

$$\mu \left| \left(\mathbf{w}(t) \cdot \nabla \mathbf{m}(t), \Psi(t) \right) \right| \leq \gamma^{-1} (\mu\rho_2 C(\Omega))^2 |\mathbf{m}(t)|^2 + \gamma \|\Psi(t)\|_1^2. \quad (68)$$

Finally,

$$\left| \left(\Theta(t) \cdot \nabla \mathbf{m}(t), \Psi(t) \right) \right| \leq C(\Omega) \|\Theta(t)\|_2 |\Psi(t)| (\|\mathbf{m}(t)\|_1 + |\mathbf{m}(t)|).$$

Since $(\mathbf{w}(t); \mathbf{m}(t)) \in B_{\mathbb{V}}^{\rho_2}$, we infer from the last estimate that

$$\begin{aligned} \mu \left| \left(\Theta(t) \cdot \nabla \mathbf{m}(t), \Psi(t) \right) \right| &\leq \mu \rho_2 C(\Omega) \|\Theta(t)\|_2 |\Psi(t)|, \\ &\leq \varkappa^{-1} (\mu \rho_2 C(\Omega)) |\Psi(t)|^2 + \varkappa \|\Theta(t)\|_2^2. \end{aligned} \quad (69)$$

Using (65)-(69) in (56) we deduce

$$\begin{aligned} \frac{1}{2} \frac{d|\Psi(t)|^2}{dt} + S \|\Psi(t)\|_1^2 &\leq 2\varkappa \|\Theta(t)\|_2^2 + 3\gamma \|\Psi(t)\|_1^2 + (\mu \rho_2 C(\Omega))^2 (10\varkappa^{-1} + \gamma^{-1}) |\Psi(t)|^2 \\ &\quad + 2\gamma^{-1} (\mu \rho_2 C(\Omega))^2 |\mathbf{m}(t)|^2. \end{aligned} \quad (70)$$

Letting $Y(t) = |\Theta(t)|^2 + |\Psi(t)|^2$ and summing up (64) and (70) side by side implies that

$$\begin{aligned} \frac{1}{2} \frac{dY(t)}{dt} + \mu_1 \|\Theta(t)\|_2^2 + S \|\Psi(t)\|_1^2 &\leq \varkappa^{-1} (\mu \rho_2 C(\Omega))^2 \|\mathbf{w}(t)\|_2^2 + 7\varkappa \|\Theta(t)\|_2^2 + 3\gamma \|\Psi(t)\|_1^2 \\ &\quad + (\mu \rho_2 C(\Omega))^2 (14\varkappa^{-1} + \gamma^{-1}) |\Psi(t)|^2 + \varkappa^{-1} (C(\Omega))^2 |\Theta(t)|^2 \\ &\quad + \varkappa^{-1} (\mu \rho_2 C(\Omega))^2 |\mathbf{w}(t)|^2 + (\mu \rho_2 C(\Omega))^2 (\varkappa^{-1} + 2\gamma^{-1}) |\mathbf{m}(t)|^2. \end{aligned} \quad (71)$$

Choosing $\varkappa = \frac{\mu_1}{14}$ and $\gamma = \frac{S}{6}$. We deduce from (71) that there exist positive constants $\tilde{C}_i, i = 1, \dots, 5$, depending only on $\Omega, \mu, S, \mu_1, \rho_2$ such that

$$\begin{aligned} \frac{1}{2} \frac{dY(t)}{dt} + \frac{\mu_1}{2} \|\Theta(t)\|_2^2 + \frac{S}{2} \|\Psi(t)\|_1^2 &\leq \tilde{C}_1 |\Theta(t)|^2 + \tilde{C}_2 |\Psi(t)|^2 + \tilde{C}_3 |\mathbf{w}(t)|^2 + \tilde{C}_4 \|\mathbf{w}(t)\|_2^2 \\ &\quad + \tilde{C}_5 |\mathbf{m}(t)|^2. \end{aligned} \quad (72)$$

Let us set $\chi_1 = 2 \max(\tilde{C}_1, \tilde{C}_2)$, $\chi_2 = 2 \max(\tilde{C}_3, \tilde{C}_4)$, and $\chi_3 = 2 \max(\tilde{C}_5, 1)$. We infer from (72) that

$$\frac{dY(t)}{dt} + \mu_1 \|\Theta(t)\|_2^2 + S \|\Psi(t)\|_1^2 \leq \chi Y(t) + \chi_2 |Z(t)|^2 + \chi_3 \|Z(t)\|_{\mathbb{V}}^2,$$

where

$$\begin{aligned} |Z(t)|^2 &= |\mathbf{w}(t)|^2 + |\mathbf{m}(t)|^2, \\ \|Z(t)\|_{\mathbb{V}}^2 &= \|\mathbf{w}(t)\|_2^2 + \|\mathbf{m}(t)\|_1^2, \\ |Z_0|^2 &= |\mathbf{w}_0|^2 + |\mathbf{m}_0|^2. \end{aligned}$$

Dropping out the second term in the left hand side of (72) and invoking Gronwall's lemma yield

$$Y(t) \leq \chi_4 e^{\chi_1 t} \int_0^t (|Z(s)|^2 + \|Z(s)\|_{\mathbb{V}}^2) ds, \forall t \geq 0,$$

where $\chi_4 = \max(\chi_2, \chi_3)$. From (46) we have

$$|Z(t)|^2 \leq |Z_0|^2 e^{\eta t},$$

and

$$\begin{aligned} \nu_0 \int_0^t \|Z(s)\|_{\mathbb{V}}^2 ds &\leq \eta_1 \int_0^t |Z(s)|^2 ds, \\ &\leq |Z_0|^2 (e^{\eta_1 t} - 1), \\ &\leq |Z_0|^2 e^{\eta_1 t}. \end{aligned}$$

Therefore

$$Y(t) \leq \chi_4 e^{(\eta_1 + \chi_1)t} |Z_0|^2 \left(\frac{1}{\eta_1} + \frac{1}{\nu_0} \right).$$

And this estimate implies that for any

$$\frac{|\mathbf{w}(t) - \mathbf{U}(t)|^2 + |\mathbf{m}(t) - \mathbf{M}(t)|^2}{|\mathbf{u}_0 - \mathbf{v}_0| + |\mathbf{b}_0 - \mathbf{c}_0|} \rightarrow 0,$$

for any $t \geq 0$ as $|\mathbf{u}_0 - \mathbf{v}_0| \rightarrow 0$ and $|\mathbf{b}_0 - \mathbf{c}_0| \rightarrow 0$. This means that the semigroup $\mathbb{S}(t) : \mathbb{H} \rightarrow \mathbb{H}$ is uniformly differentiable wrt to the initial data. \square

5. BOUNDS FOR THE FRACTAL $d_f(\mathcal{A})$ AND HAUSDORFF $d_H(\mathcal{A})$ DIMENSIONS OF \mathcal{A}

The fractal dimension of \mathcal{A} is written as $d_f(\mathcal{A})$. It is basically based on the number of closed balls of a fixed radius δ needed to cover \mathcal{A} (see, for instance, [33, Section 13.1.1]). We denote by $N(\mathcal{A}, \delta)$ the minimum number of balls in such a cover. We recall the following definition for sake of precision.

Definition 5.1 (See, for e.g., [33]). If the closure of \mathcal{A} is compact, then the fractal dimension of \mathcal{A} is given by

$$d_f(\mathcal{A}) = \limsup_{\delta \rightarrow 0} \frac{\log N(\mathcal{A}, \delta)}{\log(1/\delta)},$$

where we allow the limit in the above equation to take the value $+\infty$.

Now we recall the definition of the Hausdorff dimension of \mathcal{A} . It is based on the approximation of the d -dimensional volume of \mathcal{A} by a covering of finite number of balls $B(x_i, r_i)$ with radii $r_i \leq \delta$. Let

$$\mu(\mathcal{A}, d, \delta) = \inf \left\{ \sum_i r_i^d : r_i \leq \delta \text{ and } \mathcal{A} \subset \cup_i B(x_i, r_i) \right\}.$$

Definition 5.2. The Hausdorff dimension of \mathcal{A} is given by

$$d_H(\mathcal{A}) = \inf_{d > 0} \{d : \lim_{\delta \rightarrow 0} \mu(\mathcal{A}, d, \delta) = 0\}.$$

We have the following relation between fractal and Hausdorff dimensions.

Theorem 5.3. $d_H(\mathcal{A}) \leq d_f(\mathcal{A})$.

To calculate the bounds for the fractal dimension $d_f(\mathcal{A})$ of \mathcal{A} we will mainly follow the scheme in [33, Section 13.2]. For this purpose we let $\left\{ \Phi_i = \begin{pmatrix} \phi_i \\ \psi_i \end{pmatrix} : i = 1, 2, \dots \right\}$ be an orthonormal basis of \mathbb{H} and P_m be an orthogonal projection from \mathbb{H} onto

$$\text{Span} \left\{ \Phi_i : i = 1, 2, \dots, m \right\}.$$

For a solution $(\mathbf{u}; \mathbf{b})$ of (2) we denote by $\mathcal{L}(\mathbf{u}; \mathbf{b})$ the mapping defined by

$$\frac{d}{dt} \begin{pmatrix} \mathbf{U}(t) \\ \mathbf{M}(t) \end{pmatrix} = \mathcal{L}(\mathbf{u}; \mathbf{b}) \cdot \begin{pmatrix} \mathbf{U}(t) \\ \mathbf{M}(t) \end{pmatrix}.$$

Following the argument in [41, chapter V] we denote by $\{\Lambda_i; i \in \mathbb{N}\}$ the set of Lyapunov exponents of \mathbb{H} and we set

$$\tilde{q}_m = \lim_{t \rightarrow \infty} \sup_{(\mathbf{u}_0; \mathbf{b}_0) \in \mathcal{A}} \frac{1}{t} \int_0^t -\text{tr}(\mathcal{L}(\mathbf{u}; \mathbf{b}) \circ P_m)(s) ds.$$

It is a standard fact (see, for instance, [41, Chapter V]) that

$$\Lambda_1 + \dots + \Lambda_m \leq -\tilde{q}_m.$$

Therefore, it follows from [33, Theorem 13.16] (See also, [41, Subsection V.3.4]) that

Theorem 5.4. *If $-\tilde{q}_m < 0$, then m is the smallest positive integer such that $d_f(\mathcal{A}) \leq m$.*

The value of $\mathcal{L}(\mathbf{u}; \mathbf{b})$ at $(\phi_i, \psi_i)^T$ is defined by $\mathcal{L}(\mathbf{u}; \mathbf{b})(\phi_i, \psi_i)^T = (X; Y)$, where

$$\begin{aligned} (X, \phi) = & -\mu_1 \left(\frac{\partial \mathcal{E}(\phi_i)}{\partial x_k}, \frac{\partial \mathcal{E}(\phi)}{\partial x_k} \right) - \left(\Gamma(\mathbf{u}) \mathcal{E}(\phi_i) - \alpha \mathbf{A}_{ijkl}(\mathbf{u}(t)) \mathcal{E}(\phi_i), \mathcal{E}(\phi) \right) \\ & + \left(\mathbf{u}(t) \cdot \nabla \phi_i, \phi \right) + \left(\phi_i \cdot \nabla \mathbf{u}(t), \phi \right) + \mu \left(\psi_i \cdot \nabla \phi + \mathbf{b}(t) \cdot \nabla \phi, \psi_i \right), \end{aligned}$$

and

$$\begin{aligned} (Y, \psi) = & S(\Delta \psi_i, \psi) - \mu \left(\left(\phi_i \cdot \nabla \mathbf{b}(t), \psi \right) + \left(\mathbf{u}(t) \cdot \nabla \psi_i, \psi \right) - \left(\psi_i \cdot \nabla \mathbf{u}(t), \psi \right) \right. \\ & \left. - \left(\mathbf{b}(t) \cdot \nabla \phi_i, \psi \right) \right). \end{aligned}$$

for any $(\phi; \psi) \in \mathbb{V}$. Therefore

$$\left(\mathcal{L}(\mathbf{u}; \mathbf{b})(\phi_i; \psi_i)^T, (\phi_i; \psi_i) \right) = (X, \phi_i) + (Y, \psi_i),$$

which is equivalent to

$$\begin{aligned} \left(\mathcal{L}(\mathbf{u}; \mathbf{b})(\phi_i; \psi_i)^T, (\phi_i; \psi_i) \right) = & -\mu_1 \left(\frac{\partial \mathcal{E}(\phi_i)}{\partial x_k}, \frac{\partial \mathcal{E}(\phi_i)}{\partial x_k} \right) - \left(\Gamma(\mathbf{u}) \mathcal{E}(\phi_i) - \alpha \mathbf{A}_{ijkl} \mathcal{E}(\phi_i), \mathcal{E}(\phi_i) \right) \\ & - S|\nabla \psi_i|^2 + \left(\phi_i \cdot \nabla \mathbf{u}(t), \phi_i \right) - \mu \left(\left(\psi_i \cdot \nabla \phi_i, \mathbf{b}(t) \right) + \left(\phi_i \cdot \nabla \mathbf{b}(t), \psi_i \right) \right. \\ & \left. - \left(\psi_i \cdot \nabla \mathbf{u}(t), \psi_i \right) \right). \end{aligned}$$

Let us recall [5, Equation (2.16)]

$$\begin{aligned} I_0 = \int_{\Omega} \left(\Gamma(\mathbf{u}(t)) |\mathcal{E}(\phi_i)|^2 - \alpha \mathbf{A}_{ijkl}(\mathbf{u}(t)) |\mathcal{E}(\phi_i)|^2 \right) dx \geq & 2\varepsilon \mu_0 \int_{\Omega} \frac{|\mathcal{E}(\phi_i)|^2}{(\varepsilon + |\mathcal{E}(\mathbf{u}(t))|^2)^{1+\alpha/2}} dx \\ & + 2(1 - \alpha) \mu_0 \int_{\Omega} \frac{|\mathcal{E}(\phi_i)|^2}{(\varepsilon + |\mathcal{E}(\mathbf{u}(t))|^2)^{\alpha/2}} dx \end{aligned}$$

Let us set I_1 (resp., I_2) the first (resp., second) in the above inequality. Both of I_1 and I_2 are positive, so we have

$$\begin{aligned} - \left(\mathcal{L}(\mathbf{u}; \mathbf{b})(\phi_i; \psi_i)^T, (\phi_i; \psi_i) \right) &\geq \mu_1 \left(\frac{\partial \mathcal{E}(\phi_i)}{\partial x_k}, \frac{\partial \mathcal{E}(\phi_i)}{\partial x_k} \right) + S |\nabla \psi_i|^2 + 2(1 - \alpha) I_2 \\ &\quad - \left(\phi_i \cdot \nabla \mathbf{u}(t), \phi_i \right) - \mu \left(\left(\psi_i \cdot \nabla \phi_i, \mathbf{b}(t) \right) + \left(\phi_i \cdot \nabla \mathbf{b}(t), \psi_i \right) \right. \\ &\quad \left. - \left(\psi_i \cdot \nabla \mathbf{u}(t), \psi_i \right) \right). \end{aligned}$$

Since $0 \leq \alpha < 1$, we easily see that $p = 2 - \alpha \in (1, 2]$. We can check that

$$(2(1 - \alpha))^{-1} I_2 \geq \left(\int_{\Omega} (\varepsilon + |\mathcal{E}(\mathbf{u}(t))|^2)^{p/2} dx \right)^{\frac{p-2}{p}} \left(\int_{\Omega} |\mathcal{E}(\phi_i)|^p dx \right)^{2/p}.$$

from which along with Korn's inequality we derive that

$$(2(1 - \alpha))^{-1} I_2 \geq \left(\int_{\Omega} (\varepsilon + |\mathcal{E}(\mathbf{u}(t))|^2)^{p/2} dx \right)^{\frac{p-2}{p}} K(\Omega) \|\phi_i\|_{W^{1,p}}^2.$$

For $0 \leq \alpha < 1$ we have that

$$\int_{\Omega} (\varepsilon + |\mathcal{E}(\mathbf{u}(t))|^2)^{p/2} dx \leq C(\Omega, \alpha, \varepsilon) + \int_{\Omega} |\mathcal{E}(\mathbf{u}(t))|^p dx. \quad (73)$$

As $1 < p \leq 2$ we also see that

$$\int_{\Omega} (\varepsilon + |\mathcal{E}(\mathbf{u}(t))|^2)^{p/2} dx \leq C(\Omega, \alpha, \varepsilon) + C(\Omega) \left(\int_{\Omega} |\mathcal{E}(\mathbf{u}(t))|^2 dx \right)^{p/2} \quad (74)$$

Since $(\mathbf{u}; \mathbf{b})$ is a solution of (2), \mathbf{u} is an element of $L^2(0, \infty; \mathbb{V}_1)$. Therefore

$$\int_{\Omega} (\varepsilon + |\mathcal{E}(\mathbf{u}(t))|^2)^{p/2} dx \leq C(\Omega, \alpha, \varepsilon) + C(\Omega, \alpha),$$

for almost everywhere $t \geq 0$. Thanks to the fact that $p - 2 \geq 0$, we see that

$$(2(1 - \alpha))^{-1} I_2 \geq \left(\tilde{C}(\Omega, \alpha, \varepsilon) \right)^{\frac{p-2}{p}} K(\Omega) \|\phi_i\|_{W^{1,p}}^2,$$

where $\tilde{C}(\Omega, \alpha, \varepsilon) = C(\Omega, \alpha, \varepsilon) + C(\Omega, \alpha)$. Hence

$$\begin{aligned} - \operatorname{tr}(\mathcal{L}(\mathbf{u}; \mathbf{b}) \circ P_m) &= - \sum_{i=1}^m \left(\mathcal{L}(\mathbf{u}; \mathbf{b}) \Phi_i, \Phi_i \right) \geq \mu_1 \sum_{i=1}^m \left(\frac{\partial \mathcal{E}(\phi_i)}{\partial x_k}, \frac{\partial \mathcal{E}(\phi_i)}{\partial x_k} \right) + S \sum_{i=1}^m |\nabla \psi_i|^2 \\ &\quad + \tilde{K}(\Omega) \sum_{i=1}^m \|\phi_i\|_{W^{1,p}}^2 - \sum_{i=1}^m F(\mathbf{u}, \mathbf{b}, \phi_i, \psi_i), \end{aligned}$$

where $\tilde{K}(\Omega) = 2(1 - \alpha) \left(\tilde{C}(\Omega, \alpha, \varepsilon) \right)^{\frac{p-2}{p}}$ and

$$F(\mathbf{u}, \mathbf{b}, \phi_i, \psi_i) = (\phi_i \cdot \nabla \mathbf{u}(t), \phi_i) + \mu \left((\psi_i \cdot \nabla \phi_i, \mathbf{b}(t)) + (\phi_i \cdot \nabla \mathbf{b}(t), \psi_i) + (\psi_i \cdot \nabla \psi_i, \mathbf{u}(t)) \right).$$

From Hölder's inequality and Young's inequality we derive that

$$\begin{aligned} (\phi_i \cdot \nabla \mathbf{u}(t), \phi_i) &\leq C(\Omega) |\mathbf{u}(t)|_{L^\infty} \|\phi_i\|_{H^1}, \\ &\leq \kappa_1^{-1} C(\Omega)^2 \|\mathbf{u}(t)\|_2^2 + \kappa_1 \|\phi_i\|^2_{H^1}, \end{aligned}$$

for any $\kappa_1 > 0$. Since $|\phi_i|^2 + |\psi_i|^2 = 1$ we obtain from the last estimate that

$$(\phi_i \cdot \nabla \mathbf{u}(t), \phi_i) \leq \kappa_1^{-1} C(\Omega)^2 \|\mathbf{u}(t)\|_2^2 + \kappa_1 \|\phi_i\|_{H^1}^2.$$

By a similar argument we have that

$$\begin{aligned} \mu \left((\psi_i \cdot \nabla \phi_i, \mathbf{b}(t)) + (\phi_i \cdot \nabla \mathbf{b}(t), \psi_i) \right) &\leq 2\mu C(\Omega) \|\psi_i\|_{H^1} \|\mathbf{b}(t)\|_1 \\ &\leq \gamma^{-1} (2\mu C(\Omega))^2 \|\mathbf{b}(t)\|_1^2 + \gamma \|\phi_i\|_{H^2}^2, \end{aligned}$$

for any $\gamma > 0$. Also

$$\begin{aligned} \mu (\psi_i \cdot \nabla \psi_i, \mathbf{u}(t)) &\leq \mu C(\Omega) \|\psi_i\|_1 \|\mathbf{u}(t)\|_2 \|\psi_i\|, \\ &\leq \kappa_2^{-1} (\mu C(\Omega))^2 \|\mathbf{u}(t)\|_2^2 + \kappa_2 \|\psi_i\|_{H^1}^2, \end{aligned}$$

for any $\kappa_2 > 0$. Thus

$$\begin{aligned} F(\mathbf{u}, \mathbf{b}, \phi_i, \psi_i) &\leq \kappa_1 \|\phi_i\|_{H^1}^2 + \gamma \|\phi_i\|_{H^2}^2 + \kappa_2 \|\psi_i\|_{H^1}^2 + C(\Omega)^2 (\kappa_1^{-1} + \kappa_2^{-1} \mu^2) \|\mathbf{u}(t)\|_2^2 \\ &\quad + \gamma^{-1} (2\mu C(\Omega))^2 \|\mathbf{b}(t)\|_1^2. \end{aligned}$$

This last inequality and Korn's inequality enable us to state that

$$\begin{aligned} -\operatorname{tr}(\mathcal{L}(\mathbf{u}; \mathbf{b}) \circ P_m) &\geq \mu_1 K(\Omega) \sum_{i=1}^m \|\phi_i\|_{H^2}^2 + S \sum_{i=1}^m \|\psi_i\|_{H^1}^2 + \tilde{K}(\Omega) \sum_{i=1}^m \|\phi_i\|_{W^{1,p}}^2 \\ &\quad - \gamma \sum_{i=1}^m \|\phi_i\|_{H^2}^2 - \kappa_1 \sum_{i=1}^m \|\phi_i\|_{H^1}^2 - \gamma^{-1} (\mu C(\Omega))^2 \|\mathbf{b}(t)\|_2^2 \\ &\quad - \kappa_2 \sum_{i=1}^m \|\psi_i\|_{H^1}^2 - C(\Omega)^2 (\kappa_1^{-1} + \kappa_2^{-1} \mu^2) \|\mathbf{u}(t)\|_2^2. \end{aligned}$$

It was proved in [5] that there exists a positive constant $d(\Omega)$ such that for any $\sigma > 0$

$$|\phi|_{W^{1,p}}^2 \geq \begin{cases} \frac{1}{d(\Omega)} \|\phi\|_{H^1}^2 & \text{if } \alpha = 0, \\ \frac{\sigma^{1/\delta'}}{\delta' d(\Omega)} \|\phi\|_{H^1}^2 - \sigma^{\frac{1-\delta'}{\delta'}} \left(\frac{1-\delta'}{\delta'} \right) \|\phi\|_{H^2}^2 & \text{otherwise,} \end{cases}$$

with $\delta' = 2 \left(\frac{2-\alpha}{4+\alpha} \right)$. Therefore

$$\begin{aligned} -\operatorname{tr}(\mathcal{L}(\mathbf{u}; \mathbf{b}) \circ P_m) &\leq \left(\mu_1 K(\Omega) - \tilde{K}(\Omega) \sigma^{\frac{1-\delta'}{\delta'}} \frac{1-\delta'}{\delta'} \right) \sum_{i=1}^m \|\phi_i\|_{H^2}^2 - \gamma \sum_{i=1}^m \|\phi_i\|_{H^2}^2 \\ &\quad + \left(\tilde{K}(\Omega) \frac{\sigma^{\frac{1}{\delta'}}}{\delta' d(\Omega)} - \kappa_1 \right) \sum_{i=1}^m \|\phi_i\|_{H^1}^2 + (S - \kappa_2) \sum_{i=1}^m \|\psi_i\|_{H^1}^2 \\ &\quad - C(\Omega)^2 (\kappa_1^{-1} + \kappa_2^{-1} \mu^2) \|\mathbf{u}(t)\|_2^2 - \gamma^{-1} (4\mu C(\Omega))^2 \|\mathbf{b}(t)\|_1^2. \end{aligned}$$

As σ and γ are arbitrary, we can choose them in such a way that

$$\mu_1 K(\Omega) - \tilde{K}(\Omega) \sigma^{\frac{1-\delta'}{\delta'}} \frac{1-\delta'}{\delta'} = \frac{\mu_1 K(\Omega)}{2},$$

that is $\sigma = \left(\frac{\mu_1 K(\Omega) \delta'}{\tilde{K}(\Omega)(1 - \delta')} \right)^{\frac{\delta'}{1 - \delta'}}$ and $\gamma = \frac{\mu_1 K(\Omega)}{2}$. This choice implies that

$$\begin{aligned} -\text{tr}(\mathcal{L}(\mathbf{u}; \mathbf{b}) \circ P_m) &\geq \left[\frac{\tilde{K}(\Omega)}{\delta' d(\Omega)} \left(\frac{\mu_1 K(\Omega) \delta'}{\tilde{K}(\Omega)(1 - \delta')} \right)^{\frac{1}{1 - \delta'}} - \varkappa_1 \right] \sum_{i=1}^m \|\phi_i\|_{H^1}^2 + (S - \varkappa_2) \sum_{i=1}^m \|\psi_i\|_{H^1}^2 \\ &\quad - C(\Omega)^2 (\varkappa_1^{-1} + \varkappa_2^{-1} \mu^2) \|\mathbf{u}(t)\|_2^2 - \frac{8\mu^2}{\mu_1 K(\Omega)} \|\mathbf{b}(t)\|_1^2. \end{aligned}$$

Choosing $\varkappa_2 = S$ and $\varkappa_1 = \frac{\gamma'}{2}$, where

$$\gamma' = \frac{\tilde{K}(\Omega)}{\delta' d(\Omega)} \left(\frac{\mu_1 K(\Omega) \delta'}{\tilde{K}(\Omega)(1 - \delta')} \right)^{\frac{1}{1 - \delta'}}, \quad (75)$$

we infer from the last estimate that

$$-\text{tr}(\mathcal{L}(\mathbf{u}; \mathbf{b}) \circ P_m) \geq \frac{\gamma'}{2} \sum_{i=1}^m \|\phi_i\|_{H^1}^2 - C(\Omega)^2 \left(S^{-1} + \frac{2\mu^2}{\gamma'} \right) \|\mathbf{u}(t)\|_2^2 - \frac{8\mu^2}{\mu_1 K(\Omega)} \|\mathbf{b}(t)\|_1^2.$$

Choosing ϕ_i as the eigenfunctions of the Stokes operator whose eigenvalues satisfy

$$\lambda_j \geq \tilde{c} \lambda_1 j^n, \quad (76)$$

we derive from [33] that

$$\sum_{i=1}^m \|\phi_i\|_{H^1}^2 \geq \tilde{c} \lambda_1 m^{1+2/n}. \quad (77)$$

Thus

$$-\text{tr}(\mathcal{L}(\mathbf{u}; \mathbf{b}) \circ P_m) \geq \frac{\gamma' \tilde{c} \lambda_1}{2} m^{1+2/n} - C(\Omega)^2 \left(S^{-1} + \frac{2\mu^2}{\gamma'} \right) \|\mathbf{u}(t)\|_2^2 - \frac{8\mu^2}{\mu_1 K(\Omega)} \|\mathbf{b}(t)\|_1^2. \quad (78)$$

To give the bounds for the fractal dimension of \mathcal{A} we also need to estimate the time average of $\|\mathbf{u}(t)\|_2^2$ and $\|\mathbf{b}(t)\|_1^2$. From (13) we derive that

$$\begin{aligned} |\mathbf{u}(t)| + |\mathbf{b}(t)| + 2\mu_1 K(\Omega) \int_0^t \|\mathbf{u}(s)\|_2^2 ds + 2S \int_0^t \|\mathbf{b}(s)\|_1^2 ds &\leq 4\gamma^{-1} |f|^2 + \gamma \int_0^t |\mathbf{u}(s)|^2 ds \\ &\quad + |\mathbf{u}_0|^2 + |\mathbf{b}_0|^2. \end{aligned}$$

By using Poincaré's inequality and choosing $\gamma = \frac{\mu_1 K(\Omega)}{2\lambda_1}$ we can deduce from the above inequality that

$$\begin{aligned} \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|\mathbf{u}(s)\|_2^2 ds &\leq \frac{\Lambda}{\mu_1 K(\Omega)}, \\ \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t \|\mathbf{b}(s)\|_1^2 ds &\leq \frac{\Lambda}{S}, \end{aligned}$$

where

$$\Lambda = \frac{4\lambda_1 |f|^2}{\mu_1 K(\Omega)}. \quad (79)$$

Now we see that

$$-\tilde{q}_m \leq \left(\frac{C(\Omega)^2}{\mu_1 K(\Omega)} \left(\frac{1}{S} + \frac{2\mu^2}{\gamma'} \right) + \frac{8\mu^2}{\mu_1 K(\Omega) S} \right) \Lambda - \frac{\gamma' \tilde{c} \lambda_1}{2} m^{1+2/n}. \quad (80)$$

Owing to [33, Theorem 13.16] we see that $d_f(\mathcal{A}) \leq m$ if m is the smallest positive integer such that $-\tilde{q}_m < 0$, that is

$$m - 1 < \left[\frac{2\Lambda}{\gamma' \tilde{c} \lambda_1 \mu_1 K(\Omega)} \left(C^2(\Omega) \left(\frac{1}{S} + \frac{2\mu^2}{\gamma'} \right) + \frac{8\mu^2}{S} \right) \right]^{\frac{n}{n+2}} < m, \quad (81)$$

where γ' and Λ are given respectively by (75) and (79), and $\delta' = 2 \left(\frac{2 - \alpha}{4 + \alpha} \right)$.

The conclusion of this section is stated in the following claim that we have just proved above.

Theorem 5.5. *The global attractor \mathcal{A} for the MHD equations for the nonlinear bipolar fluids (2) is finite-dimensional with $d_H(\mathcal{A}) \leq d_f(\mathcal{A}) \leq m$ where $m > 0$ is positive integer given by*

$$m - 1 < \left[\frac{2\Lambda}{\gamma' \tilde{c} \lambda_1 \mu_1 K(\Omega)} \left(C^2(\Omega) \left(\frac{1}{S} + \frac{2\mu^2}{\gamma'} \right) + \frac{8\mu^2}{S} \right) \right]^{\frac{n}{n+2}} < m, \quad (82)$$

where γ' and Λ are given respectively by (75) and (79), and $\delta' = 2 \left(\frac{2 - \alpha}{4 + \alpha} \right)$.

ACKNOWLEDGMENT

The author's research is supported by the Austrian Science Foundation.

REFERENCES

- [1] A. V. Babin and M. I. Vishik. *Attractors of evolution equations*. Volume 25 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, 1992.
- [2] H. Bellout, F. Bloom and J. Necas. Phenomenological behavior of multipolar viscous fluids. *Quarterly of Applied Mathematics* 50:559-583, 1992.
- [3] H. Bellout, F. Bloom and J. Necas. Solutions for incompressible Non-Newtonian fluids. *C. R. Acad. Sci. Paris Sér. I. Math.* 317:795-800, 1993.
- [4] H. Bellout, F. Bloom and J. Necas. Young measure-valued solutions for Non-Newtonian incompressible fluids. *Communication in Partial Differential Equations*. 19(11& 12):1763-1803, 1994.
- [5] H. Bellout, F. Bloom and J. Necas. Bounds for the dimensions of the attractors of nonlinear bipolar viscous fluids. *Asymptotic Analysis*. 11(2):131-167, 1995.
- [6] H. Bellout, F. Bloom and J. Necas. Existence, uniqueness and stability of solutions to initial boundary value problems for bipolar fluids. *Differential and Integral Equations* 8:453-464, 1995.
- [7] D. Biskamp. *Magnetohydrodynamical Turbulence*. Cambridge University Press, Cambridge, 2003.
- [8] F. Bloom. Attractors of Non-Newtonian Fluids. *Journal of Dynamics and Differential Equations*. 7(1): 109-140, 1995.
- [9] S. Chandrasekhar. *Hydrodynamic and Hydromagnetic Stability*. Dover, 1981.
- [10] V. V. Chepyzhov and M. I. Vishik. Evolution equations and their trajectory attractors. *J. Math. Pures Appl.* (9) **76**(10): 913-964, 1997.
- [11] V. V. Chepyzhov and M. I. Vishik. Trajectory and global attractors of the three-dimensional Navier-Stokes system. *Math. Notes* 71(1-2):177-193, 2002.
- [12] V. V. Chepyzhov, M. I. Vishik, and W. L. Wendland. On non-autonomous sine-Gordon type equations with a simple global attractor and some averaging. *Discrete Contin. Dyn. Syst.* 12(1):27-38, 2005.
- [13] B. Desjardins and C. Le Bris. Remarks on a nonhomogeneous model of magnetohydrodynamics. *Differential Integral Equations*. 11(3):377-394, 1998.
- [14] Q. Du and M.D. Gunzburger. Analysis of Ladyzhenskaya Model for Incompressible Viscous Flow. *Journal of Mathematical Analysis and Applications*. **155**:21-45, 1991.
- [15] G. Duvaut and J.-L. Lions. Incompressible equations in thermoelasticity and magneto-hydrodynamics. *Arch. Rational Mech. Anal.* **46**: 241-279, 1972.
- [16] J. Freshe and M. Ruzicka. Non-homogeneous generalized Newtonian fluids. *Mathematische Zeitschrift*. **260**(2):353-375, 2008.

- [17] J.-F. Gerbeau, and C. Le Bris. Existence of solution for a density-dependent magnetohydrodynamic equation. *Adv. Differential Equations* 2(3):427-452, 1997.
- [18] J.-F. Gerbeau, C. Le Bris, and T. Lelièvre. *Mathematical methods for the Magnetohydrodynamics of Liquid Metals*. Oxford University Press, New York, 2006.
- [19] M.D. Gunzburger, O.A. Ladyzhenskaya, and J.S. Peterson. On the global unique solvability of initial-boundary value problems for the coupled modified Navier-Stokes and Maxwell equations. *J. Math. Fluid Mech.* 6(4):462-482, 2004.
- [20] M.D. Gunzburger and C. Trenchea. Analysis of an optimal control problem for the three-dimensional coupled modified Navier-Stokes and Maxwell equations. *J. Math. Anal. Appl.* 333(1):295-310, 2007.
- [21] O.A. Ladyzhenskaya and V. Solonnikov. Solution of some nonstationary magnetohydrodynamical problems for incompressible fluid. *Trudy of Steklov Math. Inst.* 69:115-173, 1960.
- [22] O. A. Ladyzhenskaya. *Mixed Boundary- Value Problem for the Hyperbolic Equation*. Moscow, 1963.
- [23] O.A. Ladyzhenskaya. *The mathematical theory of viscous incompressible flow*. Gordon and Breach, New York, 1969.
- [24] O.A. Ladyzhenskaya. New equations for the description of the viscous incompressible fluids and solvability in the large of the boundary value problems for them. In *Boundary Value Problems of Mathematical Physics V*. American Mathematical Society, Providence, RI, 1970.
- [25] O. A. Ladyzhenskaya. *Boundary-Value Problems of Mathematical Physics*, Moscow, 1973.
- [26] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*. Dunod; Gauthier-Villars, Paris, 1969.
- [27] J. Málek, J. Necas and A. Novotny. Measure-valued solutions and asymptotic behavior of a multipolar model of a boundary layer. *Czechoslovak Mathematical Journal.* 42(3):549-576, 1992.
- [28] J. Málek, J. Necas, and M. Ruzicka. On weak solutions to a class of non-Newtonian incompressible fluids in bounded three-dimensional domains: the case $p \geq 2$. *Adv. Differential Equations*
- [29] J. Málek, J. Necas, M. Rokyta and M. Ruzicka. *Weak and measure-valued solutions to evolutionary PDEs*. Applied Mathematics and Mathematical Computation, 13. Chapman & Hall, London, 1996. 6(3):257-302, 2001.
- [30] J. Necas and M. Silhavy Multipolar viscous fluids. *Quarterly of Applied Mathematics.* XLIX(2):247-266, 1991.
- [31] J. Necas, A. Novotny and M. Silhavy. Global solution to the compressible isothermal multipolar fluids. *J. Math. Anal. Appl.* 162:223-242, 1991.
- [32] P.A. Razafimandimby. Trajectory attractor for a non-autonomous Magnetohydrodynamic equations of Non-Newtonian Fluids. *Preprint, arXiv:1111.2924*, 2011.
- [33] J. C. Robinson. *Infinite-Dimensional Dynamical Systems: An Introduction to Dissipative Parabolic PDEs and the Theory of Global Attractors*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.
- [34] Samokhin, V. N. On a system of equations in the magnetohydrodynamics of nonlinearly viscous media. *Differential Equations* 27(5):628-636, 1991.
- [35] Samokhin, V. N. Existence of a solution of a modification of a system of equations of magnetohydrodynamics. *Math. USSR-Sb.* 72(2):373-385, 1992.
- [36] V. N. Samokhin. Stationary problems of the magnetohydrodynamics of non-Newtonian media. *Siberian Math. J.* 33(4):654-662, 1993.
- [37] V. N. Samokhin. The operator form and solvability of equations of the magnetohydrodynamics of nonlinearly viscous media. *Differ. Equ.* 36(6):904-910, 2000.
- [38] M. Sermange and R. Temam. Some mathematical questions related to the MHD equations. *Comm. Pure Appl. Math.* 36(5):635-664, 1983.
- [39] L. Stupyalis. An initial-boundary value problem for a system of equations of magnetohydrodynamics. *Lithuanian Math. J.* 40(2):176-196, 2000.
- [40] R. Temam. *Navier-Stokes Equations*. North-Holland, 1979.
- [41] R. Temam. *Infinite-dimensional dynamical systems in mechanics and physics*. In: Applied Mathematical Sciences, vol. 68. Springer-Verlag, New York, 1988.

E-mail address, P.A. Razafimandimby: paulrazafi@gmail.com

DEPARTMENT OF MATHEMATICS AND INFORMATION TECHNOLOGY, MONTAN UNIVERSITY LEOBEN, FRANZ JOSEF STR. 18, 8700 LEOBEN, AUSTRIA